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A nonstandard proof of a generalized demiclosedness principle

Dedicated to W. A. Kirk on the occasion of his receiving an Honorary Doctorate from Maria Curie-Skłodowska University

Abstract. Let X be a uniformly convex Banach space, C a nonempty, closed and convex subset of X and let $T : C \to X$ be an asymptotically nonexpansive in the intermediate sense mapping. In this paper we present a nonstandard proof of a demiclosedness principle for such T.

1. Introduction. Demiclosedness principle [2] is one of the basic tools in theory of nonexpansive mappings in uniformly convex Banach spaces. To state this principle we recall some definitions. The notion of uniform convexity was introduced by Clarkson in 1936 [5]. We begin with a notion of a modulus of convexity.

Definition 1.1. The modulus of convexity of a Banach space X is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}.$$

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Definition 1.2. A Banach space X is said to be uniformly convex if $\delta(\epsilon) > 0$ for each $\epsilon \in (0, 2]$.

Now we recall a few notations and notions connected with mappings. For any mapping T, we denote by $\operatorname{Fix} T$ the set of fixed points of T.

Definition 1.3. Let $C \subset X$ be a nonempty set. A mapping $T : C \to X$ is demiclosed (at y) if $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to y, then $x \in C$ and $Tx = y$.

Definition 1.4. Let $C \subset X$ be a nonempty set. A mapping $T : C \to C$ is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for $x, y \in C$.

In 1968 Browder [2] proved the following theorem (in its statement I denotes the identity mapping).

Theorem 1.1. *Suppose C is a bounded closed convex subset of a uniformly convex Banach space X and suppose $T : C \to C$ is nonexpansive. Then $I - T$ is demiclosed at* 0.

Many authors have generalized Browder's result to wider classes of mappings and Banach spaces. Now we recall three definitions of such classes of mappings. The first one is due to Goebel and Kirk [8].

Definition 1.5. Let $C \subset X$ and $T : C \to C$. If there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ as $n \to \infty$ for which

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $x, y \in C$, then T is said to be asymptotically nonexpansive.

The second is due to Kirk [13].

Definition 1.6. Let $C \subset X$ be bounded and $T : C \to C$. If T satisfies

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0$$

for each $x \in C$, and T^N is continuous for some $N \ge 1$, then T is a mapping of asymptotically nonexpansive type.

The third was introduced by Bruck, Kuczumow and Reich [3].

Definition 1.7. Let $C \subset X$ be bounded. A mapping $T : C \to C$ is called asymptotically nonexpansive in the intermediate sense if T is continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

Kirk [13] proved that if X is a uniformly convex Banach space, C is a nonempty, bounded, closed and convex subset of X and $T : C \to C$ is a mapping of asymptotically nonexpansive type, then T has a fixed point. Since self-mappings of bounded, closed and convex subsets C of a uniformly convex Banach space which are either asymptotically nonexpansive or asymptotically nonexpansive in the intermediate sense are of asymptotically nonexpansive type, they have a fixed point.

There are many results which generalize Browder's demiclosedness principle to the above classes of mappings (see, e.g., [7], [17] and the bibliography therein). A nonstandard proof of one of these results can be found in [11].

Now, we recall another definition of mapping which is asymptotically nonexpansive in the intermediate sense. In this definition a non-self-mapping $T : C \to X$ appears.

Definition 1.8 ([4]). Let X be a Banach space and $C \subset X$. Let $P : X \to C$ be a nonexpansive retraction. A mapping $T : C \to X$ is said to be asymptotically nonexpansive in the intermediate sense if T is continuous and the following inequality holds

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\| T(PT)^{n-1}x - T(PT)^{n-1}y \right\| - \|x-y\| \right) \le 0.$$

In [10], [14] and [4] the following generalized demiclosedness principles were proved. Our aim is to give nonstandard proofs of these results.

Theorem 1.2 ([10], [14]). *Let X be a uniformly convex Banach space, C a nonempty, closed and convex subset of X, and $T : C \to C$ a mapping which is asymptotically nonexpansive in the intermediate sense. If $\{x_k\}$ is a sequence in C converging weakly to $\bar{x}$ and if*

$$\lim_{j\to\infty} \left(\limsup_{k\to\infty} \left\| x_k - T^j x_k \right\| \right) = 0,$$

then $T\bar{x} = \bar{x}$.

Theorem 1.3 (Demiclosedness Principle for Non-self-mappings [4]). *Let X be a uniformly convex Banach space, C a nonempty, closed and convex subset of X, and let $P : X \to C$ be a nonexpansive retraction. Let $T : C \to X$ be a mapping which is uniformly continuous and asymptotically nonexpansive in the intermediate sense, that is,*

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\| T(PT)^{n-1}x - T(PT)^{n-1}y \right\| - \|x-y\| \right) \le 0.$$

If $\{x_k\}$ is a sequence in C converging weakly to $\bar{x}$ and if

$$\lim_{j\to\infty} \left(\limsup_{k\to\infty} \left\| x_k - T(PT)^{j-1} x_k \right\| \right) = 0,$$

then $T\bar{x} = \bar{x}$.

2. Ultrapowers. As we have mentioned, we will use nonstandard techniques. So in this section we recall a few basic facts concerning ultrapowers. The set $\mathbb{N}$ can be treated as a sequence $\{n\}_{n\in\mathbb{N}}$. Hence it has a subnet $\{n_\xi\}$ which is an ultranet (see, e.g., [9]). Throughout this section $\{n_\xi\}$ will remain fixed.

Let X be a Banach space and let

$$l_\infty(X) = \left\{ x = \{x_n\} \in X : \sup_{n\in\mathbb{N}} \|x_n\| < \infty \right\}.$$

It is known that $l_\infty(X)$ is a Banach space with the norm defined by

$$\|\{x_n\}\|_\infty = \sup_{n\in\mathbb{N}} \|x_n\|$$

for $\{x_n\} \in l_\infty(X)$. Set

$$\mathcal{N} = \left\{ \{x_n\} \in l_\infty(X) : \lim_\xi \left\|x_{n_\xi}\right\| = 0 \right\}.$$

The Banach space ultrapower $\tilde{X}$ of X (relative to the ultranet $\{n_\xi\}$) is the quotient space $l_\infty(X)/\mathcal{N}$. Thus the elements of $\tilde{X}$ consist of equivalence classes $\tilde{x} = [\{x_n\}]$ for which

$$\|\tilde{x}\|_\xi = \|[\{x_n\}]\|_\xi = \lim_\xi \left\|x_{n_\xi}\right\|,$$

with $\{u_n\} \in [\{x_n\}]$ if and only if $\lim_\xi \left\|u_{n_\xi} - x_{n_\xi}\right\| = 0$. It is known that $\tilde{X}$ with the norm $\|\cdot\|_\xi$ is a Banach space. Moreover, if X is uniformly convex, then so is $\tilde{X}$ and $\delta_{\tilde{X}} = \delta_X$. Another important fact about ultrapowers is the following result of Stern [16]: If X is super-reflexive, then $(\tilde{X})^* = \tilde{X^*}$. This means that each functional $\tilde{f} \in (\tilde{X})^*$ is of the form $\tilde{f} = [\{f_n\}] \in \tilde{X^*}$, and $\tilde{f}(\tilde{x}) = \lim_\xi f_{n_\xi}\left(x_{n_\xi}\right)$ for each $\tilde{x} \in \tilde{X}$. For more detailed description of this setting see, e.g., [1], [11], [12] and [15].

For each $x \in X$, let (x_n) denote the sequence for which $x_n \equiv x$, and let $\dot{x} = [(x_n)] \in \tilde{X}$. Then X is linearly isometric to the subspace

$$\dot{X} = \{\dot{x} : x \in X\}$$

of $\tilde{X}$ via mapping $i(x) = \dot{x}$, $x \in X$. Likewise, for $C \subset X$ we define the set $\dot{C}$. Finally, if $C \subset X$, then

$$\tilde{C} = \{\tilde{x} = [\{x_n\}] : x_n \in C \text{ for each } n\}.$$

In what follows notation introduced in this section will be used.

3. Demiclosedness principle. Here we present a nonstandard proof of the following theorem, which is a generalization of Theorem 1.3. Namely, we assume continuity of T instead of uniform continuity.

Theorem 3.1 (Demiclosedness Principle for Non-self-mappings)**.** *Let X be a uniformly convex Banach space, C a nonempty, closed and convex subset of X, and let $P : X \to C$ be a nonexpansive retraction. Let $T : C \to X$ be a mapping which is continuous and asymptotically nonexpansive in the intermediate sense, that is,*

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\| T(PT)^{n-1}x - T(PT)^{n-1}y \right\| - \|x-y\| \right) \le 0.$$

If $\{x_k\}$ is a sequence in C converging weakly to $\bar{x}$ and if

$$\lim_{j\to\infty} \left(\limsup_{k\to\infty} \left\| x_k - T(PT)^{j-1}x_k \right\| \right) = 0,$$

then $T\bar{x} = \bar{x}$.

Proof. Without loss of generality we may assume that a uniformly convex Banach space X is real. By the assumption

$$\lim_{j\to\infty} \left(\limsup_{k\to\infty} \left\| x_k - T(PT)^{j-1}x_k \right\| \right) = 0,$$

we can choose a subsequence $\{x_{k_m}\}$ of $\{x_k\}$ such that

$$\lim_{n\to\infty} \left\| x_{k_{m_n}} - T(PT)^{n-1}x_{k_{m_n}} \right\| = 0$$

and

$$\lim_{n\to\infty} \left\| x_{k_{m_n}} - T(PT)^{n}x_{k_{m_n}} \right\| = 0$$

for each subsequence $\{x_{k_{m_n}}\}$ of $\{x_{k_m}\}$. Since $\{x_{k_m}\}$ is bounded as a weakly convergent sequence, we see that $\{T(PT)^m x_{k_m}\}$ is also bounded. This implies that a sequence $\{T(PT)^n y_n\}$ is bounded for each bounded sequence $\{y_n\} \in C$. Indeed, it is sufficient to notice that by the asymptotic nonexpansiveness in the intermediate sense of T we have

$$\limsup_{n\to\infty} \left(\|T(PT)^n y_n - T(PT)^n x_{k_n}\| - \|y_n - x_{k_n}\| \right) \le 0.$$

The above observation allows us to define a mapping $S : \tilde{C} \to \tilde{X}$ by setting

$$S([\{y_n\}]) = [\{T(PT)^n y_n\}]$$

for $[\{y_n\}] \in \tilde{C}$. By the inequality

$$\limsup_{n\to\infty} \left(\|T(PT)^n y_n - T(PT)^n z_n\| - \|y_n - z_n\| \right) \le 0$$

we get

$$\begin{aligned} \|S([\{y_n\}]) - S([\{z_n\}])\|_\xi &= \lim_\xi \left\| T(PT)^{n_\xi} y_{n_\xi} - T(PT)^{n_\xi} z_{n_\xi} \right\| \\ &\le \lim_\xi \left\| y_{n_\xi} - z_{n_\xi} \right\| = \|[\{y_n\}] - [\{z_n\}]\|_\xi, \end{aligned}$$

which proves nonexpansiveness of S. Since the retraction P is nonexpansive, we see that

$$\limsup_{n\to\infty} \left(\|(PT)^n y_n - (PT)^n x_{k_n}\| - \|y_n - x_{k_n}\|\right) \leq \limsup_{n\to\infty} \left(\left\|T(PT)^{n-1} y_n - T(PT)^{n-1} x_{k_n}\right\| - \|y_n - x_{k_n}\|\right) \leq 0$$

and therefore

$$\lim_{\xi} \left\|(PT)^{n_\xi} y_{n_\xi} - (PT)^{n_\xi} z_{n_\xi}\right\| \leq \lim_{\xi} \left\|y_{n_\xi} - z_{n_\xi}\right\|.$$

So, we can introduce another nonexpansive mapping $S' : \tilde{C} \to \tilde{C}$ by setting

$$S'([\{y_n\}]) = [\{(PT)^n y_n\}]$$

for $[\{y_n\}] \in \tilde{C}$. We claim that S and S' have common fixed points. We recall that each subsequence $\{x_{k_{m_n}}\}$ of the sequence $\{x_{k_m}\}$ satisfies the following conditions

$$\lim_{n\to\infty} \left\|x_{k_{m_n}} - T(PT)^{n-1} x_{k_{m_n}}\right\| = 0$$

and

$$\lim_{n\to\infty} \left\|x_{k_{m_n}} - T(PT)^{n} x_{k_{m_n}}\right\| = 0.$$

Hence we have

$$\left\|S([\{x_{k_{m_n}}\}]) - [\{x_{k_{m_n}}\}]\right\|_\xi = \lim_{\xi} \left\|T(PT)^{n_\xi} x_{k_{m_{n_\xi}}} - x_{k_{m_{n_\xi}}}\right\| = 0$$

and

$$\begin{aligned} \|S'([\{x_{k_{m_n}}\}]) - [\{x_{k_{m_n}}\}]\|_\xi &= \left\|[\{(PT)^n x_{k_{m_n}}\}] - [\{x_{k_{m_n}}\}]\right\|_\xi \\ &= \lim_{\xi} \left\|(PT)^{n_\xi} x_{k_{m_{n_\xi}}} - x_{k_{m_{n_\xi}}}\right\| \\ &\leq \lim_{\xi} \left\|T(PT)^{n_\xi - 1} x_{k_{m_{n_\xi}}} - x_{k_{m_{n_\xi}}}\right\| = 0. \end{aligned}$$

This means that for each subsequence $\{x_{k_{m_n}}\}$ of the sequence $\{x_{k_m}\}$ the element $[\{x_{k_{m_n}}\}]$ of $\tilde{C}$ is a common fixed point of mappings S and S'. We know that $\tilde{X}$ is uniformly convex and therefore $\operatorname{Fix} S$ and $\operatorname{Fix} S'$ are closed and convex [6]. So these sets are convex and weakly closed. We claim that $\dot{\bar{x}}$ (where by assumption $\bar{x}$ is the weak limit of $\{x_{k_m}\}$) is a common element of $\operatorname{Fix} S$ and $\operatorname{Fix} S'$. If, for example, $\dot{\bar{x}} \notin \operatorname{Fix} S$ then by Separation Theorem (see Theorem 6.5 in [11]), there exists $\tilde{f} \in \tilde{X}^*$ such that

$$\tilde{f}(\dot{\bar{x}}) > \tilde{f}(\tilde{y})$$

for $\tilde{y} \in \operatorname{Fix} S$. By super-reflexivity of X there exists $f_n \in X^*$ such that for each $\tilde{u} = [\{u_n\}] \in \tilde{X}$ we have

$$\tilde{f}(\tilde{u}) = \lim_{\xi} f_{n_\xi}\left(u_{n_\xi}\right).$$

Since $\{x_{k_m}\}$ converges weakly to $\bar{x}$, we can choose a subsequence $\{x_{k_{m_n}}\}$ such that

$$\lim_{n\to\infty} \left| f_n(x_{k_{m_n}}) - f_n(\bar{x}) \right| = 0.$$

As we know, $\tilde{x} = [\{x_{k_{m_n}}\}]$ is a fixed point of S. Thus we get the following contradiction

$$\tilde{f}(\dot{\bar{x}}) > \tilde{f}(\tilde{x}) = \lim_{\xi} f_{n_\xi}\left(x_{k_{m_{n_\xi}}}\right) = \lim_{\xi} f_{n_\xi}(\bar{x}) = \tilde{f}(\dot{\bar{x}}).$$

So, $\dot{\bar{x}}$ is a common element of $\operatorname{Fix} S$ and $\operatorname{Fix} S'$. Hence

$$\bar{x} = \lim_{\xi} T(PT)^{n_\xi}\bar{x}$$

and

$$\bar{x} = \lim_{\xi} (PT)^{n_\xi}\,\bar{x}.$$

By the continuity of T, this gives

$$\bar{x} = \lim_{\xi} T(PT)^{n_\xi}\bar{x} = T\left(\lim_{\xi} (PT)^{n_\xi}\,\bar{x}\right) = T\bar{x}$$

and the proof is complete. □

It is evident that a slight change in the above proof gives a nonstandard proof of Theorem 1.2.

References

[1] Aksoy, A. G., M. A. Khamsi, *Nonstandard Methods in Fixed Point Theory*, Springer-Verlag, New York, 1990.

[2] Browder, F. E., *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.

[3] Bruck, R. E., T. Kuczumow and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property*, Colloq. Math. **LXV** (1993), 169–179.

[4] Chidume, C. E., N. Shahzad and H. Zegeye, *Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense*, Numer. Funct. Anal. Optim. **25** (2004), 239–257.

[5] Clarkson, J. A., *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

[6] DeMarr, R., *Common fixed points for commuting contraction mappings*, Pacific J. Math. **15** (1963), 1139–1141.

[7] García-Falset, J., B. Sims and M. A. Smyth, *The demiclosedness principle for mappings of asymptotically nonexpansive type*, Houston J. Math. **22** (1996), 101–108.

[8] Goebel, K., W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.

[9] Goebel K., W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.

[10] Kaczor, W., T. Kuczumow and S. Reich, *A mean ergodic theorem for mappings which are asymptotically nonexpansive in the intermediate sense*, Nonlinear Anal. **47** (2001), 2731–2742.

[11] Khamsi M. A., W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, John Willey & Sons, Inc., New York, 2001.
[12] Khamsi, M. A., B. Sims, *Ultra-methods in metric fixed point theory*, Handbook of Metric Fixed Point Theory (W. A. Kirk, B. Sims, eds.), Kluwer Academic Publishers, Dordrecht–Boston–London, 2001, pp. 177–199.
[13] Kirk, W. A., *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math. **17** (1974), 339–346.
[14] Oka, H., *An ergodic theorem for asymptotically nonexpansive mappings in the intermediate sense*, Proc. Amer. Math. Soc. **125** (1997), 1693–1703.
[15] Sims, B., *"Ultra"-techniques in Banach Space Theory*, Queen's Papers in Pure and Applied Math. 160, Queen's University, Kingston, Ontario, 1982.
[16] Stern, J., *Propriétés locales et ultrapuissances d'espaces de Banach*, Séminaire Maurey–Schwartz 1974–1975, Exp. Nos. VI et VII, École Polytech., Paris, 1975.
[17] Xu, H. K., *Existence and convergence for fixed points of mappings of asymptotically nonexpansive type*, Nonlinear Anal. **16** (1991), 1139–1146.

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