ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LIX, 2005 SECTIO A 51–65

LEOPOLD KOCZAN and PAWEŁ ZAPRAWA

On covering problems in the class of typically real functions

Abstract. Let A be a class of analytic functions on the unit disk Δ. In this article we extend the concept of the Koebe set and the covering set for the class A. Namely, for a given $D \subset \Delta$ the plane sets of the form

$$\bigcap_{f \in A} f(D) \quad \text{and} \quad \bigcup_{f \in A} f(D)$$

we define to be the Koebe set and the covering set for the class A over the set D. For any A and $D = \Delta$ we get the usual notion of Koebe and covering sets. In the case $A = T$, the normalized class of typically real functions, we describe the Koebe domain and the covering domain over disks $\{z : |z| < r\} \subset \Delta$ and over the lens-shaped domain $H = \{z : |z + i| < \sqrt{2}\} \cap \{z : |z - i| < \sqrt{2}\}$.

Introduction. Let $\mathcal{A}$ be the family of all analytic functions f on the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$, $A \subset \mathcal{A}$ and let D be a subdomain of Δ with $0 \in D$. The plane sets $K_A(D) = \bigcap_{f \in A} f(D)$, $L_A(D) = \bigcup_{f \in A} f(D)$, $K_A = K_A(\Delta)$ and $L_A = L_A(\Delta)$ we shall call the Koebe domain for the class A over the set D, the covering domain for the class A over the set D, the Koebe domain for the class A and the covering domain for the class A, respectively. Except some special cases, the sets $K_A(D)$ are open connected and hence domains. Note that for

2000 *Mathematics Subject Classification.* Primary 30C45, Secondary 30C75.
Key words and phrases. Typically real functions, Koebe domain, covering domain.

$$A = \left\{ z \mapsto \frac{1}{a}(e^{az} - 1) : a \in \mathbf{C} \setminus \{0\} \right\}$$

and

$$B = \left\{ z \mapsto \frac{1}{2n} \left[\left(\frac{1+z}{1-z} \right)^n - 1 \right] : n = m, m+1, m+2, \dots \right\},$$

$m > 0$, we have $K_A = \{0\}$, and hence $K_{\mathcal{A}} = \{0\}$, and the sets K_B are not open. For many important classes, the Koebe domains were discussed in a number of papers and some sharp results are well known (see [4] for more details).

The determination of sets K_A and L_A is usually more difficult if the considered classes are not rotation invariant, which means that the following property

$$f \in A \Leftrightarrow e^{-i\varphi} f(ze^{i\varphi}) \in A \quad \text{for any} \quad \varphi \in \mathbf{R} \tag{1}$$

is not satisfied.

For instance, (1) is not satisfied by each nontrivial class A with real coefficients. One of them is the class T of typically real functions, i.e. functions $f \in \mathcal{A}$ and satisfying the condition

$$\operatorname{Im} z \operatorname{Im} f(z) \geq 0 \quad \text{for} \quad z \in \Delta\,.$$

The Koebe domain for the class T was found by Goodman [3].

Theorem A (Goodman)**.** *The Koebe domain for the class T is symmetric with respect to both axes, and the boundary of this domain in the upper half plane is given by the polar equation*

$$\varrho(\theta) = \begin{cases} \frac{\pi \sin\theta}{4\theta(\pi-\theta)} & \text{for } \theta \in (0, \pi), \\ \frac{1}{4} & \text{for } \theta = 0 \text{ or } \theta = \pi. \end{cases}$$

The covering domain for the class T is the whole plane because for members $f_1(z) = \frac{z}{(1-z)^2}$ and $f_{-1}(z) = \frac{z}{(1+z)^2}$ we have $f_1(\Delta) \cup f_{-1}(\Delta) = \mathbf{C}$.

Clearly, each time if $\{f_1, f_{-1}\} \subset A \subset \mathcal{A}$ then $\mathbf{C}$ is the covering domain for A. However, for many classes the covering domain may give some interesting information (like for classes of bounded functions).

Basic properties of $K_A(D)$ and $L_A(D)$ established in the following two theorems are easy to prove.

First, let us denote by ∂D the boundary of a set D. Moreover, we use the notation:

$$\Delta_r = \{z \in \mathbf{C} : |z| < r\},$$
$$S = \{f \in \mathcal{A} : f \text{ is univalent in } \Delta\},$$
$$\mathcal{A}R = \{f \in \mathcal{A} : f \text{ has real coefficients}\}.$$

Theorem 1. *For a fixed class $A \subset \mathcal{A}$, the following properties of $K_A(D)$ are true:*

1. *if A satisfies (1) and $A \subset S$, then $K_A(\Delta_r) = \Delta_{m(r)}$, where $m(r) = \min\{|f(z)| : f \in A, z \in \partial\Delta_r\}$;*
2. *if $A \subset \mathcal{A}R$ and D is symmetric with respect to the real axis, then $K_A(D)$ is symmetric with respect to the real axis;*
3. *if $A \subset \mathcal{A}R$ consists of only such f that $-f(-z) \in A$, and if D is symmetric with respect to both axes, then $K_A(D)$ is symmetric with respect to both axes;*
4. *if $D_1 \subset D_2$, then $K_A(D_1) \subset K_A(D_2)$;*
5. *if $A_1, A_2 \subset \mathcal{A}$ and $A_1 \subset A_2$, then $K_{A_2}(D) \subset K_{A_1}(D)$.*

Theorem 2. *For a fixed class $A \subset \mathcal{A}$, the following properties of $L_A(D)$ are true:*

1. *if A satisfies (1) and $A \subset S$, then $L_A(\Delta_r) = \Delta_{M(r)}$, where $M(r) = \max\{|f(z)| : f \in A, z \in \partial\Delta_r\}$;*
2. *if $A \subset \mathcal{A}R$ and D is symmetric with respect to the real axis, then $L_A(D)$ is symmetric with respect to the real axis;*
3. *if $A \subset \mathcal{A}R$ consists of only such f that $-f(-z) \in A$, and if D is symmetric with respect to both axes, then $L_A(D)$ is symmetric with respect to both axes;*
4. *if $D_1 \subset D_2$, then $L_A(D_1) \subset L_A(D_2)$;*
5. *if $A_1, A_2 \subset \mathcal{A}$ and $A_1 \subset A_2$, then $L_{A_1}(D) \subset L_{A_2}(D)$.*

In accordance with simple results concerning the known classes S, ST, CV and CC consisting of normalized univalent, starlike, convex and close-to-convex functions respectively, we have

$$K_S(\Delta_r) = K_{ST}(\Delta_r) = K_{CC}(\Delta_r) = \Delta_{m(r)},$$

where $m(r) = \frac{r}{(1+r)^2}$, $r \in (0, 1]$,

$$K_{CV}(\Delta_r) = \Delta_{m(r)},$$

where $m(r) = \frac{r}{1+r}$, $r \in (0, 1]$,

$$L_S(\Delta_r) = L_{ST}(\Delta_r) = L_{CC}(\Delta_r) = \Delta_{M(r)},$$

where $M(r) = \frac{r}{(1-r)^2}$, $r \in (0, 1)$,

$$L_{CV}(\Delta_r) = \Delta_{M(r)},$$

where $M(r) = \frac{r}{1-r}$, $r \in (0, 1)$,

$$L_S(\Delta) = L_{ST}(\Delta) = L_{CC}(\Delta) = L_{CV}(\Delta) = \mathbf{C}.$$

In this paper we determine Koebe domains and covering domains for the class T over some special sets, like disks Δ_r and the lense-shaped domain $H = \{z : |z+i| < \sqrt{2}\} \cap \{z : |z-i| < \sqrt{2}\}$.

Covering domains $L_T(D)$. First of all, let us consider the case $D = \Delta_r$, where $r \in (0,1)$. Since the class T does not satisfy (1), the set $L_T(\Delta_r)$ is not equal to $\Delta_{M(r)}$, where $M(r) = \max\{|f(z)| : f \in T, z \in \partial\Delta_r\} = \frac{r}{(1-r)^2}$, but is a proper subset of $\Delta_{M(r)}$.

Denote

$$f_t(z) = \frac{z}{1-2zt+z^2}\ ,\ t \in [-1,1]. \tag{2}$$

These functions are univalent, starlike in the unit disk and

$$ET = \{f_t(z) : t \in [-1,1]\}, \tag{3}$$

where ET means the set of extreme points of the class T (see for example [5]). The following lemma is true for the functions of the form (2).

Lemma 1. *For $t \in [0,1]$ we have $f_{-t}(\Delta_r) \cap \{w : \operatorname{Re} w > 0\} \subset f_t(\Delta_r) \cap \{w : \operatorname{Re} w > 0\}$.*

Proof. The above inclusion is true for $t = 0$. Let $t \in (0,1]$. If $1/f_t(z) = 1/f_{-t}(\zeta) = u + iv$ and $|z| = |\zeta| = r$ then

$$\left(\frac{u+2t}{r+1/r}\right)^2 + \left(\frac{v}{1/r-r}\right)^2 = 1 = \left(\frac{u-2t}{r+1/r}\right)^2 + \left(\frac{v}{1/r-r}\right)^2,$$

i.e. $u = 0$, $v^2 = [(1+r^2)^2 - 4t^2r^2](1-r^2)^2/(r+r^3)^2$, $z = \frac{2tr^2}{1+r^2} - i\frac{vr^2}{1-r^2}$, $\zeta = -\overline{z}$. Thus

$$f_t(\partial\Delta_r) \cap f_{-t}(\partial\Delta_r) = \{i\varrho, -i\varrho\}\ ,$$

where

$$\varrho = 1/|v| = (r+r^3)/[(1-r^2)\sqrt{(1+r^2)^2 - 4t^2r^2}].$$

The inequality $f_t(r) > f_{-t}(r)$ completes the proof. □

By the Robertson formula for the class T, the set $\{f(z) : f \in T\}$ is the closed convex hull of the circular arc $\{f_t(z) : -1 \le t \le 1\}$, so we have [2]:

Theorem B (Goluzin). *Let $z = re^{i\varphi} \in \Delta \setminus \{0\}$, $0 < \varphi < \pi$ and $R = r/[2(1-r^2)\sin\varphi$. The set $\{f(z) : f \in T\}$ is the closed convex segment bounded by the arc $\{f_t(z) : -1 \le t \le 1\}$ and the line segment joining the points $f_1(z)$, $f_{-1}(z)$. Clearly, $\{f_t(z) : -1 \le t \le 1\} \subset \{w :\ |w - iR| = R\}$.*

One can obtain from this theorem that the upper estimate of the set of moduli of typically real functions in a fixed point $z \in \Delta$ is attained by the functions of the form (2). The lower estimation is attained by a suitable function of the form

$$f = \alpha f_1 + (1-\alpha)f_{-1}\ ,\ \alpha \in [0,1]. \tag{4}$$

Let r be an arbitrary fixed number in $(0,1)$.

Theorem 3. $L_T(\Delta_r) = f_1(\Delta_r) \cup f_{-1}(\Delta_r)$.

Proof. The property 3 from Theorem 2 gives that the covering domain $L_T(\Delta_r)$ is symmetric with respect to both coordinate axes. It suffices to determine the boundary of this set only in the first quadrant of $\mathbf{C}$ plane.

To do this, we discuss

$$\max\{|f(z)| : f \in T,\ |z| = r,\ \arg f(z) = \alpha\}\ ,\ \alpha \in [0, \tfrac{\pi}{2}]\,. \tag{5}$$

According to Theorem B, we have

$$\begin{aligned}&\max\{|f(z)| : f \in T,\ |z| = r,\ \arg f(z) = \alpha\}\\ &\qquad = \max\{|f_t(z)| : t \in [-1,1],\ |z| = r,\ \arg f_t(z) = \alpha\}.\end{aligned} \tag{6}$$

Clearly, the maximum of the right hand side of (6) is obtained by some f_{t_0} if and only if the minimum

$$\min\left\{\tfrac{1}{4}|f_t(z)|^{-2} : t \in [-1,1],\ |z| = r,\ \arg f_t(z) = \alpha\right\} \tag{7}$$

is obtained also by f_{t_0}.

According to Lemma 1 we discuss $t \in [0,1]$ only.

Denote by $h(t,\varphi)$ the function we are minimizing, i.e.

$$h(t,\varphi) = \frac{1}{4}|f_t(re^{i\varphi})|^{-2} = \frac{1}{4}\left|re^{i\varphi} + \frac{1}{r}e^{-i\varphi} - 2t\right|^2 = t^2 - 2at\cos\varphi + a^2 - \sin^2\varphi,$$

with $a = \frac{1}{2}(r + \frac{1}{r}) > 1$.

Since the function $\varphi \mapsto \Gamma(\varphi) = \sin\varphi/(a\cos\varphi - t)$ strictly increases on intervals of the domain of Γ, the condition $\arg f_t(re^{i\varphi}) = \alpha$ can be written as follows:

$$\frac{\sqrt{a^2-1}\sin\varphi}{a\cos\varphi - t} = \tan\alpha \quad \text{for} \quad 0 < \varphi < \arccos\left(\frac{t}{a}\right) \tag{8}$$

and

$$0 = \alpha \quad \text{for} \quad \varphi = 0\ , \quad \frac{\pi}{2} = \alpha \quad \text{for} \quad \varphi = \arccos\left(\frac{t}{a}\right) \le \frac{\pi}{2}. \tag{9}$$

Let $0 < \alpha < \frac{\pi}{2}$. We are going to prove that the minimum of h on the curve (8) is attained outside of the set $\{(t,\varphi)\ :\ 0 < t < 1\ ,\ 0 < \varphi < \arccos\left(\frac{t}{a}\right)\}$. On the contrary, if there existed an (t_0,φ_0), $0 < t_0 < 1$, $0 < \varphi_0 < \arccos\left(\frac{t_0}{a}\right)$, which realizes the minimum (7), then there would be a Lagrange function

$$H(t,\varphi) \equiv h(t,\varphi) - \lambda\left[\frac{\sqrt{a^2-1}\sin\varphi}{a\cos\varphi - t} - \tan\alpha\right]$$

such that $\frac{\partial H}{\partial t}(t_0,\varphi_0) = \frac{\partial H}{\partial\varphi}(t_0,\varphi_0) = 0$ and $\sqrt{a^2-1}\sin\varphi_0/(a\cos\varphi_0 - t_0) = \tan\alpha$. Reducing λ from the above system of equalities we get

$$\left[(t_0 - a\cos\varphi_0)^2 + (a^2-1)\sin^2\varphi_0\right]\cos\varphi_0 = 0,$$

a contradiction. Thus (7) is equal to

$$\min\left\{\frac{1}{4}|f_t(re^{i\varphi})|^{-2} : t(1-t)=0,\ 0<\varphi<\arccos\left(\frac{t}{a}\right),\ \arg f_t(re^{i\varphi})=\alpha\right\}.$$

But $0<\varphi<\frac{\pi}{2}$, $\sqrt{a^2-1}\sin\varphi/a\cos\varphi=\tan\alpha$ implies

$$\sin\varphi = a\sin\alpha/\sqrt{a^2-\cos^2\alpha}\in(0,1) \quad\text{and}\quad \frac{1}{4}|f_0(re^{i\varphi})|^{-2}=\frac{a^2(a^2-1)}{a^2-\cos^2\alpha}.$$

Similarly, if $0<\varphi<\arccos\left(\frac{1}{a}\right)$ and $\frac{\sqrt{a^2-1}\sin\varphi}{a\cos\varphi-1}=\tan\alpha$, then

$$\cos\varphi=\frac{1+a\cos\alpha}{a+\cos\alpha}\in\left(\frac{1}{a},1\right)$$

and

$$\frac{1}{4}|f_1(re^{i\varphi})|^{-2}=\left(\frac{a^2-1}{a+\cos\alpha}\right)^2<\frac{a^2(a^2-1)}{a^2-\cos^2\alpha}.$$

Thus $|f_0(re^{i\varphi_0})|<|f_1(re^{i\varphi_1})|$ for

$$0<\varphi_0<\frac{\pi}{2}\ ,\ 0<\varphi_1<\arccos\left(\frac{1}{a}\right)\ ,\ \arg f_0(re^{i\varphi_0})=\arg f_1(re^{i\varphi_1})=\alpha.$$

In particular,

$$\max\{|f(z)| : f\in T,\ |z|=r,\ \arg f(z)=\alpha\}$$
$$=\left|f_1\left(\frac{r(1+a\cos\alpha+i\sqrt{a^2-1}\sin\alpha)}{a+\cos\alpha}\right)\right|.$$

Finally, we should examine two cases: $\alpha=0$ and $\alpha=\frac{\pi}{2}$. For $\alpha=0$ we have $h(t,0)=(a-t)^2\geq h(1,0)$. In the case $\alpha=\frac{\pi}{2}$ we obtain

$$h(t,\varphi)=(a^2-1)\left(1-\frac{t^2}{a^2}\right)\geq h(1,\varphi).$$

It means that for every function $f\in T$

$$f(\Delta_r)\cap\{w:\operatorname{Re}w\geq 0\}\subset f_1(\Delta_r)\cap\{w:\operatorname{Re}w\geq 0\}.$$

From the equation $f_{-t}(-z)=-f_t(z)$, which is true for the functions of the form (2), we consequently have

$$f(\Delta_r)\cap\{w:\operatorname{Re}w\leq 0\}\subset f_{-1}(\Delta_r)\cap\{w:\operatorname{Re}w\leq 0\}.$$

□

From Theorem 3 we conclude:

Corollary 1. *For every function $f\in T$ and $z\in\partial\Delta_r$ (i.e. $|z|=r$) we have*

1. $|f(z)|\leq\frac{r}{(1-r)^2}$,
2. $|\operatorname{Re}f(z)|\leq\frac{r}{(1-r)^2}$,

3. $|\operatorname{Im} f(z)| \le \frac{\sqrt{2[(1+r^2)\sqrt{1+34r^2+r^4}-1+14r^2-r^4](\sqrt{1+34r^2+r^4}+1+r^2)}}{8[(1+r^2)\sqrt{1+34r^2+r^4}+1-14r^2+r^4]}$.

Observe that Theorem 3 still holds for $r = 1$.

As it was said, the set $L_T(\Delta)$ is the whole complex plane $\mathbf{C}$. It is easy to see that Δ could be replaced by another set for which the covering domain is still the whole plane.

Let us consider the lens-shaped domain H. For $z \in \partial H$ we have $|z+\frac{1}{z}| = 2$ and hence $z + \frac{1}{z} = 2e^{i\varphi}$, $\varphi \in (-\pi, \pi]$. Therefore, the boundary of the image of H under the function f_1 is a straight line $\operatorname{Re} w = -\frac{1}{4}$ because $f_1(z) = \frac{1}{2(e^{i\varphi}-1)} = -\frac{1}{4}(1 + i\cot\frac{\varphi}{2})$. It implies that $f_1(H) = \{w \in \mathbf{C} : \operatorname{Re} w > -\frac{1}{4}\}$. Likewise, it could be shown that $f_{-1}(H) = \{w \in \mathbf{C} : \operatorname{Re} w < \frac{1}{4}\}$. We have proved:

Theorem 4. $L_T(H) = \mathbf{C}$.

The plain question appears: are there other sets $D \subset H$, $D \neq H$ such that $L_T(D) = \mathbf{C}$ or, is there the smallest set D_0 having this property (in the sense that $L_T(D_0) = \mathbf{C}$ and whose every proper subset D satisfies $L_T(D) \neq \mathbf{C}$)?

Let us denote by E_a the subset of Δ such that $z+\frac{1}{z}$ belongs to the exterior of an ellipse $u = 2\cos\tau$, $v = 2a\sin\tau$, where $a \ge 1$, $\tau \in (-\pi, \pi]$. Hence

$$E_a = \left\{ z \in \Delta : \left| z + \frac{1}{z} + 2i\sqrt{a^2-1} \right| + \left| z + \frac{1}{z} - 2i\sqrt{a^2-1} \right| > 4a \right\}.$$

In special case $E_1 = H$.

For $z \in \partial E_a \cap \{z : \operatorname{Im} z > 0\}$ or equivalently $z+\frac{1}{z} = 2(\cos\tau + ia\sin\tau)$, $\tau \in (-\pi, 0)$ we have

$$f_1(z) = -\frac{1}{4[1+(a^2-1)\cos^2\frac{\tau}{2}]}\left(1 + ia\cot\frac{\tau}{2}\right)$$

and

$$f_{-1}(z) = \frac{1}{4[1+(a^2-1)\sin^2\frac{\tau}{2}]}\left(1 - ia\tan\frac{\tau}{2}\right).$$

This yields that $f_1(E_a) \supset \{w : \operatorname{Re} w \ge 0\}$ and $f_{-1}(E_a) \supset \{w : \operatorname{Re} w \le 0\}$, and eventually $f_1(E_a) \cup f_{-1}(E_a) = \mathbf{C}$. This could be written in the form:

Theorem 5. *For every $a \ge 1$ we have $L_T(E_a) = \mathbf{C}$.*

Observe that $E_\infty = \lim_{a\to\infty} E_a$ is not a domain, and it consists of two disjoined domains H_1 and H_{-1} given by

$$H_1 = \left\{ z \in \Delta : \operatorname{Re}\left(z + \frac{1}{z}\right) > 2 \right\} \quad \text{and}$$
$$H_{-1} = \left\{ z \in \Delta : \operatorname{Re}\left(z + \frac{1}{z}\right) < -2 \right\}. \tag{10}$$

These sets appear in the known property of typically real functions [2], [6]:

(11)
$$\begin{aligned} &\forall f \in T \quad |f_{-1}(z)| \le |f(z)| \le |f_1(z)| \ \text{ for } \ z \in H_1 \quad \text{and} \\ &\forall f \in T \quad |f_1(z)| \le |f(z)| \le |f_{-1}(z)| \ \text{ for } \ z \in H_{-1}. \end{aligned}$$

The image of the curve ∂H_1 under f_1 coincides with the imaginary axis, as well as the image of the curve ∂H_{-1} under f_{-1}. Consequently, $f_1(H_1) = \{w : \operatorname{Re} w > 0\}$ and $f_{-1}(H_{-1}) = \{w : \operatorname{Re} w < 0\}$.

It is known that these two functions attain the upper and the lower estimate of argument of typically real functions [2]. For this reason there is no function $f \in T$ for which

$$\begin{aligned} |\arg f(z)| &\le |\arg f_1(z)| = \frac{\pi}{2} \ \text{ for } \ z \in \partial H_1 \quad \text{and} \\ |\arg f(z)| &\ge |\arg f_{-1}(z)| = \frac{\pi}{2} \ \text{ for } \ z \in \partial H_{-1}. \end{aligned}$$

This leads to the conclusion:

Theorem 6.

$$L_T(H_1 \cup H_{-1}) = \mathbf{C} \setminus \{it : t \in \mathbf{R}\}, \quad L_T(\mathrm{cl}(H_1 \cup H_{-1})) = \mathbf{C},$$

where $\mathrm{cl}(A)$ *stands for a closure of a set* A.

This theorem provides that the set $\mathrm{cl}(H_1 \cup H_{-1})$ is the smallest set having the covering set equal to the whole plane (because there does not exist a set $D \subset \mathrm{cl}(H_1 \cup H_{-1})$, $D \neq \mathrm{cl}(H_1 \cup H_{-1})$ such that $L_T(D) = \mathbf{C}$).

In the above presented results we have found a covering set over a given set $D \subset \Delta$. One can research these domains from another angle. Assume that Ω is a covering domain over some domain D. Our aim is to find D.

This problem is easy to solve when $\Omega = \Delta_M$. If $L_T(D) = \Delta_M$, $M > 0$, then every boundary point of Δ_M is attained by some function of the form (2). Certainly, both statements are equivalent: $|f_t(z)| < M$, $t \in [-1, 1]$ and $|z + \frac{1}{z} - 2t| > \frac{1}{M}$, $t \in [-1, 1]$, which we can rewrite as a system of conditions

$$\begin{aligned} \left| z + \frac{1}{z} + 2 \right| &> \frac{1}{M} \ \text{ for } z \in \Delta \ , \ \operatorname{Re}\left(z + \frac{1}{z} \right) < -2 \ , \\ \left| \operatorname{Im}\left(z + \frac{1}{z} \right) \right| &> \frac{1}{M} \ \text{ for } z \in \Delta \ , \ \left| \operatorname{Re}\left(z + \frac{1}{z} \right) \right| \le 2 \ , \\ \left| z + \frac{1}{z} - 2 \right| &> \frac{1}{M} \ \text{ for } z \in \Delta \ , \ \operatorname{Re}\left(z + \frac{1}{z} \right) > 2 \ . \end{aligned}$$

Let us denote by D_M, $M > 0$ the set

$$\left\{z \in \Delta : \left|z + \frac{1}{z} - 2\right| > \frac{1}{M},\ \mathrm{Re}\left(z + \frac{1}{z}\right) > 2\right\} \cup \left\{z \in \Delta : \left|z + \frac{1}{z} + 2\right| > \frac{1}{M},\ \mathrm{Re}\left(z + \frac{1}{z}\right) < -2\right\} \cup \left\{z \in \Delta : \left|\mathrm{Im}\left(z + \frac{1}{z}\right)\right| > \frac{1}{M},\ \left|\mathrm{Re}\left(z + \frac{1}{z}\right)\right| \le 2\right\}.$$

Using the introduced notation we have

$$D_M = \left\{z \in H_1 : |z-1|^2 > \frac{1}{M}|z|\right\} \cup \left\{z \in H_{-1} : |z+1|^2 > \frac{1}{M}|z|\right\} \cup \left\{z \in \Delta \setminus (H_1 \cup H_{-1}) : \left|\mathrm{Im}\left(z + \frac{1}{z}\right)\right| > \frac{1}{M}\right\}.$$

Then

Theorem 7. $L_T(D_M) = \Delta_M$.

Koebe domains $K_T(D)$. The minimum of modulus of typically real functions for a fixed $z \in \Delta$ is attained by the functions of the form (4), which are not univalent (except for f_1 and f_{-1}). It means that calculating this minimum in all directions $e^{i\alpha}$ is not the same as finding the Koebe domain. This is the reason why the determination of Koebe domains for the class T is usually more difficult than the determination of covering domains. According to Goodman [3], the boundary of the Koebe domain over Δ consists of the images of points on the unit circle under infinite-valent functions that are called the universal typically real functions.

We will avoid the problem of not univalent functions if we consider the Koebe domain over the lens-shaped domain H and over disks Δ_r with sufficiently small radius (i.e. $r \le \sqrt{2} - 1$).

Theorem 8. $K_T(H) = \Delta_{\frac{1}{4}}$.

Proof. Set $\Gamma = \partial H \setminus \{-1, 1\}$, $\Gamma_+ = \{z \in \Gamma : \mathrm{Im}\, z > 0\}$, $\Gamma_- = \{z \in \Gamma : \mathrm{Im}\, z < 0\}$. We shall find the envelope of the family of line segments $\{\alpha f_1(z) + (1-\alpha) f_{-1}(z) : 0 < \alpha < 1\}$, $z \in \Gamma$.

Let $z \in \Gamma_+$ which is the same as $z + \frac{1}{z} = 2e^{i\varphi}$, $\varphi \in (-\pi, 0)$. The complex parametric equation of each line segment connecting $f_1(z)$ and $f_{-1}(z)$ is as follows

$$w(t) = \frac{1}{2(e^{i\varphi} - 1)} + t\left[\frac{1}{2(e^{i\varphi} + 1)} - \frac{1}{2(e^{i\varphi} - 1)}\right],\ t \in [0,1],\ \varphi \in (-\pi, 0),$$

and the real parametric equation is of the form

$$\begin{cases} x(t) = -\frac{1}{4} + \frac{1}{2}t \\ y(t) = -\frac{1}{4}\cot\frac{\varphi}{2} + \frac{1}{2}t\cot\varphi\ ,\ t \in [0,1]\ ,\ \varphi \in (-\pi, 0). \end{cases}$$

Hence, we have one parameter family of segments given by $y = -\frac{1}{4}\cot\frac{\varphi}{2} + (x + \frac{1}{4})\cot\varphi$, where $x \in [-\frac{1}{4}, \frac{1}{4}]$.

Reducing φ from the system

$$\begin{cases} y = -\frac{1}{4}\cot\frac{\varphi}{2} + \left(x + \frac{1}{4}\right)\cot\varphi \\ 0 = \frac{1}{8}\frac{1}{\sin^2\frac{\varphi}{2}} - \left(x + \frac{1}{4}\right)\frac{1}{\sin^2\varphi}\ , \end{cases}$$

we obtain the envelope of this family satisfying the equation $x^2 + y^2 = \frac{1}{16}$. Since $x \in [-\frac{1}{4}, \frac{1}{4}]$, we conclude that $\partial\Delta_{\frac{1}{4}} \cap \{w : \operatorname{Im} w > 0\}$ is the investigated envelope. Clearly, the envelope of this family for $z \in \Gamma_-$ is $\partial\Delta_{\frac{1}{4}} \cap \{w : \operatorname{Im} w < 0\}$.

From the above and from Theorem B it follows that for a fixed $z \in \Gamma$:

$$\{f(z) : f \in T\} \cap \Delta_{\frac{1}{4}} = \emptyset \quad \Rightarrow \quad \forall_{f\in T}\ f(\Gamma) \cap \Delta_{\frac{1}{4}} = \emptyset$$
$$\Rightarrow \quad \forall_{f\in T}\ \Delta_{\frac{1}{4}} \subset f(H) \quad \Rightarrow \quad \Delta_{\frac{1}{4}} \subset K_T(H).$$

All typically real functions are univalent in H, see [3], hence for any $f \in T$ we have $f(\Gamma) \subset \partial f(H)$. It means that for an arbitrary point w, $|w| = \frac{1}{4}$, there exists the only one function $f \in T$ such that $w \in \partial f(H)$. It is that function of the form (4) for which the segment $[f_{-1}(z), f_1(z)]$ is tangent to the derived envelope for all $z \in \Gamma$. Hence $K_T(H) \subset \Delta_{\frac{1}{4}}$. □

Remark. The relation $K_T(H) \subset \Delta_{\frac{1}{4}}$ can be proved in another way. One can check that

$$\partial\Delta_{\frac{1}{4}} \cap \{w : \operatorname{Im} w \geq 0\} = \{\alpha f_1(z_\alpha) + (1-\alpha)f_{-1}(z_\alpha)\ :\ \alpha \in [0,1]\},$$

where z_α is the only solution of $\alpha f_1'(z) + (1-\alpha)f_{-1}'(z) = 0$ in the set $\Delta \cap \{z : \operatorname{Im} z \geq 0\}$.

From the property 4 of Theorem 1 it follows that the set $K_T(H) = \Delta_{\frac{1}{4}}$ is contained in $K_T(\Delta)$. Theorem 8 states that $K_T(H) \neq K_T(\Delta)$. Both domains have only two common boundary points $z = 1$ and $z = -1$. Let us recall the known result of Brannan and Kirwan [1]:

Theorem C (Brannan, Kirwan)**.** *If $f \in T$, then $\Delta_{\frac{1}{4}} \subset f(\Delta)$.*

We can improve this result as follows.

Theorem 9. *If $f \in T$, then $\Delta_{\frac{1}{4}} \subset f(H)$.*

Moreover, we can establish more general version of Theorem 8 concerning sets E_a, $a > 1$.

Theorem 10. *For any $a \geq 1$ the set $K_T(E_a)$ is the convex domain having the boundary curve of the form $16x^2 + \frac{4(1+a^2)^2}{a^2}y^2 = 1$.*

Proof. Let $z \in \partial E_a \cap \{z : \operatorname{Im} z > 0\}$. Then $z + \frac{1}{z} = 2(\cos\tau + ia\sin\tau)$, $\tau \in (-\pi, 0)$. The line segment connecting $f_1(z)$ and $f_{-1}(z)$ is given by the complex parametric equation

$$w(t) = \frac{-1}{2\sin\tau(\tan\frac{\tau}{2} - ai)} + t\frac{1}{\sin^2\tau(\cot\frac{\tau}{2} + ai)(\tan\frac{\tau}{2} - ai)}\ ,\ t \in [0,1]\ ,$$

or by the real parametric equation

$$\begin{cases} x(t) = -\frac{1}{4[1+(a^2-1)\cos^2\frac{\tau}{2}]} + t\frac{1+a^2}{4[1+(a^2-1)\sin^2\frac{\tau}{2}][1+(a^2-1)\cos^2\frac{\tau}{2}]} \\ y(t) = -\frac{-a}{4\tan\frac{\tau}{2}[1+(a^2-1)\cos^2\frac{\tau}{2}]} + t\frac{2a\cot\frac{\tau}{2}}{4[1+(a^2-1)\sin^2\frac{\tau}{2}][1+(a^2-1)\cos^2\frac{\tau}{2}]}\ . \end{cases}$$

After simple calculation we can write the equation of one parameter family of line segments

$$2ax\cos\tau - (1+a^2)y\sin\tau - \frac{a}{2} = 0.$$

From the system

$$\begin{cases} 2ax\cos\tau - (1+a^2)y\sin\tau - \dfrac{a}{2} = 0 \\ -\,2ax\sin\tau - (1+a^2)y\cos\tau = 0 \end{cases}$$

one can obtain the equation of envelope

$$16x^2 + \frac{4(1+a^2)^2}{a^2}y^2 = 1\ . \tag{12}$$

Since $t \in [0,1]$ is equivalent to $x \in [-\frac{1}{4}, \frac{1}{4}]$, we conclude that whole curve (12) is the envelope of the considered family of line segments.

From the convexity of the set $16x^2 + \frac{4(1+a^2)^2}{a^2}y^2 < 1$, from univalence of all typically real functions in each E_a , $a \geq 1$ (because $E_a \subset H$) and the argument similar to that given in the proof of Theorem 8 we obtain $K_T(E_a) = \left\{(x,y) :\ 16x^2 + \frac{4(1+a^2)^2}{a^2}y^2 < 1\right\}$. □

Corollary 2. $K_T(E_\infty) = \emptyset$.

The above presented method of determining an envelope is also suitable for $a \in (0,1)$. In this case, sets E_a contain H, the domain of univalence and local univalence for the class T. Therefore, envelopes obtained in this way do not coincide with the boundary curves of the Koebe domains over E_a , $a \in (0,1)$. From the equation (12) we know that the sets bounded by these envelopes, which can be written as $L_T(E_a)\setminus L_T(\partial E_a)$, are contained in $\Delta_{\frac{1}{4}}$. It means that the presented method of envelopes fails for determining the sets E_a , $0 < a < 1$.

Finally, let us consider the Koebe domains over Δ_r, $r \in (0, \sqrt{2}-1]$. The method of an envelope is still good for deriving $K_T(\Delta_r)$. Similarly to the argument given above, this method works for any $r \in (0,1)$, but an envelope obtained in this way would be the boundary of the Koebe domain only for such a disk, in which all typically real functions (among other functions (4), too) are univalent. It holds only for $r \le \sqrt{2}-1$.

For a fixed $r \in (0,1]$ we use the notation

$$w_{-1}(\varphi) = f_{-1}(re^{i\varphi}) \quad , \quad w_1(\varphi) = f_1(re^{i\varphi})\,,$$

$$v(\varphi) = \left[\frac{\cos\varphi}{2(r+\frac{1}{r})} + \frac{(\frac{1}{r}-r)^2\sin^2\varphi\cos\varphi}{(r+\frac{1}{r})(r^2+\frac{1}{r^2}-2\cos 2\varphi)}\right] + i\frac{(\frac{1}{r}-r)\sin^3\varphi}{r^2+\frac{1}{r^2}-2\cos 2\varphi}$$

and

$$w(\varphi) = \begin{cases} w_{-1}(\varphi)\ , & \varphi \in [0, \varphi_0(r)]\ , \\ v(\varphi)\ , & \varphi \in (\varphi_0(r), \frac{\pi}{2}]\ , \end{cases} \tag{13}$$

where $\varphi_0(r) = \arccos\frac{1}{4}[\sqrt{(r+\frac{1}{r})^2+32} - (r+\frac{1}{r})]$.

From now on we make the assumption:

$$\arg w_{-1}(0) = 0 \quad , \quad \arg v(0) = 0 \quad , \quad \arg w_1(0) = 0\,,$$
$$\arg\,[w_1(0) - w_{-1}(0)] = 0 \quad , \quad \arg w'_{-1}(0) = \tfrac{\pi}{2}\,.$$

Theorem 11. *The domain $K_T(\Delta_r)$ for $r \in (0, \sqrt{2}-1]$ is symmetric with respect to both axes with $w = 0$ belonging to it. Its boundary in the first quadrant of the complex plane is the curve of the form $w([0, \frac{\pi}{2}])$.*

The proof is based on the following four lemmas.

Lemma 2. *The function* $\arg w'_{-1}(\varphi)$

1. *is increasing in $[0,\pi]$ for $r \in (0, 2-\sqrt{3}]$,*
2. *is decreasing in $[0, \varphi_1(r)]$ and is increasing in $[\varphi_1(r), \pi]$ for $r \in (2-\sqrt{3}, 1]$,*

where $\varphi_1(r) = \arccos\frac{1+r^2}{4r}$.

Proof. Let $h(\varphi) = \big(\arg w'_{-1}(\varphi)\big)'$. We have

$$h(\varphi) = \mathrm{Re}\left(1 + re^{i\varphi}\frac{f''_{-1}(re^{i\varphi})}{f'_{-1}(re^{i\varphi})}\right) = \mathrm{Re}\,\frac{1-4re^{i\varphi}+r^2e^{2i\varphi}}{1-r^2e^{2i\varphi}}$$
$$= \frac{1}{|1-r^2e^{2i\varphi}|^2}(1-r^2)(1-4r\cos\varphi+r^2)\ .$$

For $r \in (0, 2-\sqrt{3}]$ the function h is positive for all $\varphi \in [0,\pi]$, and for $r \in (2-\sqrt{3}, 1]$ the function h is negative in $[0, \varphi_1(r))$ and positive in $(\varphi_1(r), \pi]$. $\square$

Lemma 3. *For $\varphi \in [0, \varphi_0(r))$ we have*

$$\arg w'_{-1}(\varphi) - \arg\left[w_1(\varphi) - w_{-1}(\varphi)\right] > 0 \ .$$

Proof. Let $h(\varphi) = \arg w'_{-1}(\varphi) - \arg\left[w_1(\varphi) - w_{-1}(\varphi)\right]$. Then

$$\begin{aligned} h(\varphi) &= \arg\left[\frac{1-z}{(1+z)^3} iz\right] - \arg\left[\frac{z}{(1-z)^2} - \frac{z}{(1+z)^2}\right] \\ &= \arg\frac{1-z}{1+z} - \arg\frac{z}{(1-z)^2} + \frac{\pi}{2} \\ &= \frac{\pi}{2} - \left[\arctan\frac{2r\sin\varphi}{1-r^2} + \arctan\frac{(1-r^2)\sin\varphi}{(1+r^2)\cos\varphi - 2r^2}\right] \ . \end{aligned}$$

From the equation $h(\varphi) = 0$ it follows that $2\cos^2\varphi + (r + \frac{1}{r})\cos\varphi - 4 = 0$. Therefore, $\varphi = \varphi_0(r)$ is the only solution of $h(\varphi) = 0$ in $[0, \frac{\pi}{2}]$. Since $h(0) > 0$, so $h(\varphi) > 0$ for $\varphi \in [0, \varphi_0(r))$. $\square$

Lemma 4. *The envelope of the family of line segments $[w_{-1}(\varphi), w_1(\varphi)]$, where $\varphi \in (0,\pi)$, coincides with $v([\varphi_0(r), \pi - \varphi_0(r)])$.*

Proof. We begin with calculating the envelope of the family of straight lines containing these segments. We have an equation of these lines:

$$x\left(\frac{1}{r^2} - r^2\right)\sin 2\varphi + y\left[2 - \left(\frac{1}{r^2} + r^2\right)\cos 2\varphi\right] = \left(\frac{1}{r} - r\right)\sin\varphi \ .$$

From

$$\begin{cases} x\left(\frac{1}{r^2} - r^2\right)\sin 2\varphi + y\left[2 - \left(\frac{1}{r^2} + r^2\right)\cos 2\varphi\right] - \left(\frac{1}{r} - r\right)\sin\varphi = 0 \\ 2x\left(\frac{1}{r^2} - r^2\right)\cos 2\varphi + 2y\left(\frac{1}{r^2} + r^2\right)\sin 2\varphi - \left(\frac{1}{r} - r\right)\cos\varphi = 0 \ . \end{cases}$$

we obtain the envelope which can be written in the form $w = v(\varphi)$, $\varphi \in (0,\pi)$, where v is defined by (12). This curve is regular because $(\operatorname{Re} v'(\varphi))^2 + (\operatorname{Im} v'(\varphi))^2 \neq 0$, which can be concluded from the fact that the system

$$\begin{cases} \operatorname{Re} v'(\varphi) = 0 \\ \operatorname{Im} v'(\varphi) = 0 \end{cases}$$

has no solution for $\varphi \in (0,\pi)$.

Moreover, observe

$$\arg\left[w_1(\varphi) - w_{-1}(\varphi)\right] = 2\arg\frac{re^{i\varphi}}{1 - r^2 e^{2i\varphi}} \ ,$$

hence starlikeness of the function $z \to \frac{z}{1-z^2}$ implies that the argument of the tangent vector to the curve $v\left((0,\pi)\right)$ is increasing.

The envelope of the family of line segments is constructed of these points of $v\left((0,\pi)\right)$ for which

$$\arg w_{-1}(\varphi)\le \arg v(\varphi)\le \arg w_1(\varphi)$$

or equivalently

$$\operatorname{Im} w_{-1}(\varphi)\le \operatorname{Im} v(\varphi)\le \operatorname{Im} w_1(\varphi)\quad \text{for}\quad \varphi\in(0,\tfrac{\pi}{2}]$$

and

$$\operatorname{Im} w_{-1}(\varphi)\ge \operatorname{Im} v(\varphi)\ge \operatorname{Im} w_1(\varphi)\quad \text{for}\quad \varphi\in[\tfrac{\pi}{2},\pi)\ .$$

For $\varphi\in(0,\pi)$ we have

$$\frac{1}{r+\frac{1}{r}+2|\cos\varphi|}\le\frac{\sin^2\varphi}{r+\frac{1}{r}-2|\cos\varphi|}\ ,$$

and hence

$$2\cos^2\varphi+\left(r+\frac{1}{r}\right)\cos\varphi-4\le 0\ ,$$

and finally

$$\varphi\in[\varphi_0(r),\pi-\varphi_0(r)]\ .$$

We have proved that the envelope of the family of line segments $[w_{-1}(\varphi),w_1(\varphi)]$ and the curve $v([\varphi_0(r),\pi-\varphi_0(r)])$ are the same. □

Let A_φ , $\varphi\in(0,\frac{\pi}{2}]$ be the sector given by

$$A_\varphi=\{u\in\mathbf{C}:\arg w_{-1}(\varphi)\le\arg\left[u-w_{-1}(\varphi)\right]\le\arg\left[w_1(\varphi)-w_{-1}(\varphi)\right]\}$$

and let

$$l_1=\{u\in\mathbf{C}:\arg u=\arg w_{-1}(\varphi)\}\ ,$$
$$l_2=\{u\in\mathbf{C}:\arg u=\arg\left[w_1(\varphi)-w_{-1}(\varphi)\right]\}\ .$$

Denote by E the domain which is bounded, symmetric with respect to both axes and whose boundary in the first quadrant of the complex plane is identical with $w([0,\frac{\pi}{2}])$.

Lemma 5. *For $\varphi\in[0,\frac{\pi}{2}]$ we have*

1. $E\cap A_\varphi=\emptyset$,
2. $\operatorname{cl}(E)\cap A_\varphi$ *is a one-point set.*

Proof. Observe that from Lemma 2 the curve $w([0,\frac{\pi}{2}])$ has only one inflexion point $w(\varphi_1)$ when $r\in(2-\sqrt{3},(\sqrt{24}-\sqrt{15})/3)$ and $w(\varphi_0)$ if $r\in((\sqrt{24}-\sqrt{15})/3,\sqrt{2}-1]$.

Let us discuss the case $r\in(2-\sqrt{3},(\sqrt{24}-\sqrt{15})/3)$. Let $\varphi\in(0,\varphi_1(r)]$. From Lemma 2, Lemma 3 and monotonicity of $\arg\left[w_1(\varphi)-w_{-1}(\varphi)\right]$ we conclude

$$\begin{aligned}A_\varphi&\subset\{u\in\mathbf{C}:\arg w_{-1}(\varphi)\le\arg\left[u-w_{-1}(\varphi)\right]\le\arg\left[w_1(\varphi_1)-w_{-1}(\varphi_1)\right]\}\\&\subset\{u\in\mathbf{C}:\arg w_{-1}(\varphi)\le\arg\left[u-w_{-1}(\varphi)\right]\le\arg w'_{-1}(\varphi_1)\}\ .\end{aligned}$$

It means that $A_\varphi \cap f_{-1}(\Delta_r) = \emptyset$ and hence $A_\varphi \cap E = \emptyset$, since $E \subset f_{-1}(\Delta_r)$. Let $\varphi \in (\varphi_1(r), \varphi_0(r)]$. From Lemma 2 and Lemma 3 we have

$$A_\varphi \subset \{u \in \mathbf{C} : \arg w_{-1}(\varphi) \le \arg [u - w_{-1}(\varphi)] \le \arg w'_{-1}(\varphi)\} .$$

It means that $A_\varphi \cap f_{-1}(\Delta_r) = \emptyset$ and hence $A_\varphi \cap E = \emptyset$, since $E \subset f_{-1}(\Delta_r)$. Furthermore, $\mathrm{cl}(E) \cap A_\varphi = w_{-1}(\varphi)$ for $\varphi \in (0, \varphi_0]$.
Let $\varphi \in (\varphi_0, \frac{\pi}{2}]$. Then l_2 is tangent to $w((\varphi_0, \frac{\pi}{2}))$. From starlikeness of f_{-1} and the definition of E it follows that $w_{-1}(\varphi) \notin E$ and consequently $A_\varphi \cap E = \emptyset$. The sets $\mathrm{cl}(E)$ and A_φ have only one common point, i.e. the tangential point.

In the case $r \in (0, 2 - \sqrt{3})$ and $r \in (\sqrt{24} - \sqrt{15})/3, \sqrt{2} - 1]$ lemma can be proved slightly more easily, proceeding analogously to the case proven above, dividing the segment $[0, \frac{\pi}{2}]$ into two $[0, \varphi_0(r)]$ and $(\varphi_0(r), \frac{\pi}{2}]$. □

Proof of Theorem 11. Let $r \in (0, \sqrt{2} - 1]$. From Lemma 5 it follows that $E \subset K_T(\Delta_r)$. The definition of the Koebe domain leads to

$$K_T(\Delta_r) \subset \bigcap_{\alpha \in [0,1]} (\alpha f_1 + (1 - \alpha) f_{-1})(\Delta_r) = E .$$

Hence $K_T(\Delta_r) = E$. □

References

[1] Brannan, D. A., W. E. Kirwan, *A covering theorem for typically real functions*, Glasg. Math. J. **10** (1969), 153–155.

[2] Goluzin, G., *On typically real functions*, Mat. Sb. **27** (69) (1950), 201–218.

[3] Goodman, A. W., *The domain covered by a typically real function*, Proc. Amer. Math. Soc. **64** (1977), 233–237.

[4] Goodman, A. W., *Univalent Functions*, Mariner Pub. Co., Tampa, 1983.

[5] Hallenbeck, D. J., T. H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Pitman Adv. Publ. Program, Boston–London–Melbourne, 1984.

Leopold Koczan
Department of Applied Mathematics
Lublin University of Technology
ul. Nadbystrzycka 38D
20-618 Lublin, Poland
e-mail: l.koczan@pollub.pl

Paweł Zaprawa
Department of Applied Mathematics
Lublin University of Technology
ul. Nadbystrzycka 38D
20-618 Lublin, Poland
e-mail: p.zaprawa@pollub.pl

Received October 4, 2004