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**On the perfectness of groups  
of diffeomorphisms with no restriction  
on support**

ABSTRACT. It is well known that the compactly supported identity component of the group of all  $C^r$ -diffeomorphisms of a smooth manifold is perfect and simple provided  $1 \leq r \leq \infty$ ,  $r \neq n + 1$ , where  $n$  is the dimension of the manifold. Several generalizations for the automorphism groups of geometric structures are known. The problem of the perfectness of analogous groups with no restriction on support is studied. By making use of deformation principles we investigate under what conditions diffeomorphism groups are perfect provided so are their compactly supported subgroups.

**1. Introduction.** A group  $G$  is called perfect if  $G = [G, G]$ , where the commutator subgroup  $[G, G]$  is generated by all commutators  $[f, g] = f g f^{-1} g^{-1}$ ,  $f, g \in G$ . In terms of homology of groups this means that  $H_1(G) = G/[G, G] = 0$ . Observe that any nonabelian simple group is perfect.

As a converse statement for homeomorphism groups of a manifold can be regarded a theorem of Epstein [6]. It says that for a ‘typical’ transitive group of homeomorphisms  $G$  the commutator subgroup  $[G, G]$  is simple. But this theorem works only for compactly supported groups.

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From now on by  $M$  we denote the interior of a compact topological manifold  $\bar{M}$ . It follows that the set of ends of  $M$  is finite. If  $M$  is a  $C^r$ -smooth manifold as above,  $\text{Diff}^r(M)$ ,  $r = 1, \dots, \infty$ , will stand for the group of all  $C^r$ -diffeomorphisms of  $M$ . We endow  $\text{Diff}^r(M)$  with the compact-open  $C^r$ -topology. For  $f \in \text{Diff}^r(M)$  we define  $\text{supp}(f) = \overline{\{x : f(x) \neq x\}}$  and  $\text{Diff}_c^r(M) = \{f \in \text{Diff}^r(M) : \text{supp}(f) \text{ compact}\}$ . By  $\text{Diff}^r(M)_0$  (resp.  $\text{Diff}_c^r(M)_0$ ) we denote the identity component of  $\text{Diff}^r(M)$  (resp.  $\text{Diff}_c^r(M)$ ).

If  $\partial\bar{M} = \emptyset$  then it is well known that  $\text{Diff}^r(M)_0$  is a simple group, except possibly  $r = 1 + \dim(M)$ , and that the identity component of the homeomorphisms group  $\mathcal{H}(M)_0$  is simple as well (cf. [11], [5], [23], [12], [7]). In the sequel we will assume that  $\partial\bar{M} \neq \emptyset$ .

Throughout the subscript  $c$  will denote the compactly supported subgroup and the subscript  $0$  the identity component in the relevant topology.

The aim of this note is to study the problem of perfectness of diffeomorphism groups on open manifolds. In particular, we wish to reveal possible connections between the perfectness (or the first homology group) of such groups and the perfectness (or the first homology group) of their compactly supported subgroups. On the next page we give a list of selected results on the problem.

On the table the mark of interrogation indicates that the problem is open, and the sign plus indicates that the problem can be answered in the affirmative by arguments presented in this note. The symbol  $H_1(G_c)$  in the volume form and symplectic case indicates that  $H_1(G_c)$  is expressed by means of some invariants. In the first row a fragmentation property for homeomorphisms (Corollary 3.1 in [5]) is used. In the seventh row Fukui's result [8] was originally formulated for  $M = \mathbb{R}^n$  but it is easy to obtain it for an arbitrary  $M$ . Note that McDuff in [15], [16] showed also that  $H_1(G) = 0$  for  $M = \mathbb{R}^n$ , provided  $n \geq 3$ . Concerning the case of the contactomorphism group let us formulate the following conjecture: If  $G = \text{Diff}^\infty(M, \alpha)_0$ , where  $\alpha$  is a contact form, then  $H_1(G_c) = 0$  and  $H_1(G) = 0$ . It seems likely that the equality  $H_1(G) = 0$  would be proved by a refinement of methods presented here. Concerning the two last rows we conjecture also that  $H_1(G) = 0$  but a possible proof is unclear because of the nontransitivity of the groups in question.

The results in the second column are deep and usually difficult; they were proved by various methods. By a well-known Thurston–Mather isomorphism these results are related to the connectivity of the classifying space of the corresponding Haefliger structure.

the group $G$	result for $G_c$	result for $G$
$G = \mathcal{H}(M)_0$	$H_1(G_c) = 0$ , Mather [11]	$H_1(G) = 0$ McDuff [14]
$G = \mathcal{H}_{\text{Lip}}(M)_0$ Lipschitz homeomorphisms	$H_1(G_c) = 0$ Abe–Fukui [1]	+
$G = \text{Diff}^r(M)_0$ , $r \neq n+1$ $r$ finite, $n = \dim M$	$H_1(G) = 0$ Mather [12]	$H_1(G) = 0$ McDuff [14]
$G = \text{Diff}^{n+1}(M)_0$	?	$H_1(G) = 0$ McDuff [14]
$G = \text{Diff}^1(\mathbb{R}^n)_0$	$H_k(G_c) = 0$ , $\forall k \geq 1$ Tsuboi [24]	?
$G = \text{Diff}^\infty(M)_0$	$H_1(G_c) = 0$ Herman [10], Thurston [23], Epstein [7]	$H_1(G) = 0$ McDuff [14]
$G = \text{Diff}^\infty(M, \{pt\})_0$ diffeomorphisms fixing point and orientation	$H_1(G_c) = \mathbb{R}$ Fukui [8]	+
$G = \text{Diff}^\infty(M, \Omega)_0$ $\Omega$ a volume form	$H_1(G_c)$ Thurston [22]	$H_1(G)$ McDuff [15], [16]
$G = \text{Diff}^\infty(M, \omega)_0$ $\omega$ a symplectic form	$H_1(G_c)$ Banyaga [2]	?
$G = \text{Diff}^\infty(M, \alpha)_0$ $\alpha$ a contact form	?	?
$G = \text{Diff}^\infty(M, \mathcal{F})_0$ diffeomorphisms preserving leaves of a foliation $\mathcal{F}$	$H_1(G_c) = 0$ Rybicki [18]	?
$G = \text{Diff}^\infty(\mathbb{R}^m, \Lambda)_0$ diffeomorphisms preserving the product Poisson structure $\Lambda$	$H_1(G_c) = 0$ Rybicki [19]	?

In this note we exploit a fundamental paper of Segal [20] concerning the classifying spaces for foliations. In view of [20] there is only a very loose relation between the perfectness of a diffeomorphism group and the perfectness of its compactly supported subgroup. In Theorem 5.3 we formulate

conditions under which a diffeomorphism group with no restriction on support is perfect. The proof follows Segal's argument. In the last section we give simple examples of groups to which the theorem applies. Of course Segal's methods could be applied in more complicated reasonings as well. As an example can serve a beautiful paper of McDuff [16] on the group of volume preserving diffeomorphisms.

**2. Milnor's join.** Let  $G$  be a topological group. For  $G$  we may define Milnor's join  $EG$ , and  $BG = EG/G$ , the classifying space for principal  $G$ -bundles.

More precisely, let  $EG = G * G * \dots$  be a join of infinite copies of  $G$ . By definition any element of  $EG$  has the form

$$\langle g, t \rangle = (t_0 g_0, t_1 g_1, \dots, t_n g_n, \dots),$$

where  $g_i \in G$ ,  $t_i \in [0, 1]$ ,  $i = 0, 1, \dots$ , such that  $\sum_{i=0}^{\infty} t_i = 1$  and only finite number of  $t_i$  do not vanish. Two points  $\langle g, t \rangle$  and  $\langle g', t' \rangle$  are identified if and only if  $t_i = t'_i$  for all  $i$  and  $g_i = g'_i$  for all  $i$  such that  $t_i = t'_i > 0$ . The group  $G$  acts from the right on  $EG$  by  $\langle g, t \rangle h = \langle gh, t \rangle$ ,  $h \in G$ .

$EG$  is endowed with the initial topology of the mappings  $t_i : EG \ni \langle x, t \rangle \mapsto t_i \in [0, 1]$  and  $x_i : t_i^{-1}(0, 1] \ni \langle g, t \rangle \mapsto g_i \in G$ , where  $i = 0, 1, \dots$

In view of the obvious equalities  $x_i(\langle g, t \rangle h) = x_i(\langle g, t \rangle)h$  and  $t_i(\langle g, t \rangle h) = t_i(\langle g, t \rangle)$  the action  $EG \times G \rightarrow EG$  is continuous. Thus  $EG$  is a  $G$ -space. Let  $BG = EG/G$  and let  $p : EG \rightarrow BG$  be the canonical projection.

The following result of Milnor is well known.

**Theorem 2.1.** *( $EG, p, BG$ ) is a numerical principal  $G$ -bundle. Moreover, it is a universal bundle for principal  $G$ -bundles, i.e. for any paracompact space  $B$  there is a bijective correspondence between isomorphism classes of principal  $G$ -bundles over  $B$  and homotopy classes of continuous mappings  $B \rightarrow BG$ .*

For our purposes we need the following well-known facts.

**Proposition 2.2** ([4]). *If  $G$  is a discrete group then  $H_1(BG, \mathbb{Z}) = G/[G, G]$ .*

**Proposition 2.3** ([25]). *Let  $f_1, f_2 : G \rightarrow G$  be automorphisms of a discrete group  $G$  such that for every set  $g_1, \dots, g_k \in G$  there is  $h \in G$  such that  $f_1(g_i) = h f_2(g_i) h^{-1}$  for  $i = 1, \dots, k$ . Then the induced automorphisms*

$$f_{1*}, f_{2*} : H_*(BG, \mathbb{R}) \rightarrow H_*(BG, \mathbb{R})$$

*are equal.*

**3. Deformation principles.** The paper of Siebenmann [21] gives a powerful method for deforming homeomorphisms on topological manifolds (see also [5]). Moreover, this method is generalized for so-called CS-sets (locally cone-like stratified sets). Some results in this note could be also extended

to homeomorphism groups of CS-sets, but we will not study this problem here.

**Definition 3.1.** Let  $M$  be a manifold (as in Section 1), and let  $G(M) \subset \mathcal{H}(M)$  or  $G(M) \subset \text{Diff}^r(M)$ ,  $r = 1, \dots, \infty$ , be a group of homeomorphisms or diffeomorphisms, respectively. We say that  $G(M)$  fulfils the *deformation principle* if for  $A \subset A'$  closed subsets of  $M$  such that  $A'$  is a neighbourhood of  $A$  and for any compact subsets  $B, K \subset M$  such that  $B \subset \text{Int} K$ , the following statement always holds:

If  $h \in G(M)$  is equal to  $\text{id}$  on  $A'$  and  $h$  is sufficiently near  $\text{id}$  then there is an isotopy  $h_t$ ,  $0 \leq t \leq 1$ , in  $G(M)$  such that  $h_0 = h$  on  $M$ ,  $h_1 = \text{id}$  on  $A \cup B$  and  $h_t = h$  on  $A$  and on  $M \setminus K$ . Furthermore, the isotopy  $h_t$  is a continuous function of  $h$ , and  $h_t = \text{id}$  if  $h = \text{id}$ .

**Theorem 3.2** ([21]).  $\mathcal{H}(M)_0$  and  $\text{Diff}^r(M)_0$ ,  $r = 1, \dots, \infty$ , satisfy the *deformation principle*.

An easy consequence of Definition 3.1 is the following

**Corollary 3.3** (Fragmentation property). *If  $G(M)$  fulfils the deformation principle then  $G_c(M)_0$  satisfies the fragmentation property, i.e. if  $h$  is a homeomorphism and  $\text{supp}(h) \subset U_1 \cup \dots \cup U_r$ , where  $U_i$  is an open ball, then there are  $h_i$ ,  $i = 1, \dots, s$ , such that  $\text{supp}(h_i) \subset U_{j(i)}$  and  $h = h_1 \dots h_s$ .*

**Corollary 3.4.** *Suppose that  $G(M)$  satisfies the deformation principle. If  $B$  and  $K$  are compact subsets of  $M$  such that  $B \subset \text{Int} K$  then for any  $h \in G(M)$  sufficiently close to the identity there is  $g \in G(M)$  such that  $g = h$  on  $B$  and  $\text{supp}(g) \subset K$ .*

Indeed, under the notation of Definition 3.1 put  $g = h_1^{-1}h$ .

**Corollary 3.5.** *Under the above assumption, let  $B$  be a compact subset of  $M$  and let*

$$C = \bigcup C_i$$

*be the union of a countable locally finite family of pairwise disjoint compact subsets  $C_i$  such that  $B \cap C = \emptyset$ . For any  $h \in G(M)$  sufficiently close to the identity such that  $\text{supp}(h) \subset B \cup C$  there is  $g \in G(M)$  such that  $g = h$  on  $B$  and  $\text{supp}(g) \subset B$ .*

**Proof.** In fact, take a compact set  $K$  with  $B \subset \text{Int} K$  and  $K \cap C = \emptyset$ . Denote  $A = \overline{M \setminus (B \cup C)}$ . By assumption there is a closed neighborhood  $A'$  of  $A$  such that  $h = \text{id}$  on  $A'$ . Therefore, by using Definition 3.1,  $g = h_1^{-1}h$  satisfies the claim.  $\square$

Theorem 3.2 may be specified to its foliated version.

**Theorem 3.6.** *Let  $(M, \mathcal{F})$  be a topological or smooth foliated manifold. Let  $\mathcal{H}(M, \mathcal{F})$  (resp.  $\text{Diff}^r(M, \mathcal{F})$ ) denote the group of all leaf preserving homeomorphisms (resp. diffeomorphisms) of  $(M, \mathcal{F})$ . Then the groups  $\mathcal{H}(M, \mathcal{F})_0$  and  $\text{Diff}^r(M, \mathcal{F})_0$  fulfil the deformation principle.*

The proof is straightforward in the smooth case. In the topological case one can proceed as in [9] by making use of difficult methods of [5].

**4. Segal's reasoning.** In this section all groups are considered with the discrete topology.

Segal [20] considered the manifold  $X = Z \times [0, \infty)$ , where  $Z$  is a compact manifold, and the group

$$\text{Diff}^r(X, \text{rel } Z \times \{0\}) := \{h \in \text{Diff}^r(X) : h = \text{id near } Z \times \{0\}\}.$$

He proved that this group is acyclic, i.e. all reduced homology group vanish, where  $r = 1, \dots, \infty$ .

Let  $\mathcal{S}$  be the set of all positive sequences of  $\mathbb{R}$  increasing to  $\infty$ . Let  $G$  be a subgroup of  $\text{Diff}^r(X, \text{rel } Z \times \{0\})$  fulfilling the deformation principle. For  $S \in \mathcal{S}$  we define

$$G_S = \{h \in G : h = \text{id in a neighbourhood of } Z \times S\}.$$

Then one has:

- (1)  $G_S \subset G_T$  if and only if  $T \supset S$ .
- (2)  $G_S \cap G_T = G_{S \cup T}$ .

Define a homomorphism  $\Sigma : G_S \rightarrow G_c^S := \{g : S \rightarrow G_c\}$  by  $h \mapsto (s \mapsto h|_{Z \times [0, s)})$ . Here the restriction  $h|_{Z \times [0, s)}$  is understood as equal to the identity off  $Z \times [0, s)$ . Observe that such an element belongs to  $G_c$ . In fact, if  $h \in G_S$  then we may and do assume that  $h$  is sufficiently close to the identity, since  $G_S$  is a topological group. Fix  $s = s_k \in S$  and take compact subsets  $B, K$  such that  $\text{supp}(h) \cap (Z \times [0, s)) \subset \text{Int } B \subset B \subset \text{Int } K \subset Z \times [0, s)$  and use Corollary 3.5 with  $C = \bigcup_{i=0}^{\infty} C_i$  such that

$$\text{supp}(h) \cap (Z \times (s_{k+i}, s_{k+i+1})) \subset \text{Int}(C_i) \subset C_i \subset Z \times (s_{k+i}, s_{k+i+1}).$$

**Lemma 4.1** ([20]). *If  $G = \text{Diff}^r(X, \text{rel } Z \times \{0\})$ , then  $\Sigma : G_S \rightarrow G_c^S$  is a homology equivalence for all  $S \in \mathcal{S}$ .*

In the next section we will sketch Segal's proof of this lemma in a slightly more general context.

Then in view of the commutative diagram

$$\begin{array}{ccc} G_{S \cup T} & \longrightarrow & G_S \times G_T \\ \downarrow \Sigma & & \downarrow \Sigma \times \Sigma \\ G_c^{S \cup T} & \xrightarrow{\cong} & G_c^S \times G_c^T \end{array}$$

and of Lemma 4.1 we obtain that  $BG_{S \cup T} \rightarrow BG_S \times BG_T$  is a homology equivalence whenever  $S \cap T = \emptyset$ .

Let  $B_0G := \bigcup_{S \in \mathcal{S}} BG_S$ . We wish to show that  $B_0G$  is acyclic. It follows from a direct-limit argument that we need only to show that for any finite sequence  $S_1, \dots, S_m \in \mathcal{S}$  the inclusion  $\bigcup_{i=1}^m BG_{S_i} \hookrightarrow B_0G$  induces the zero map on the reduced homology level.

We obtain this by using the Mayer–Vietoris sequence and the induction (cf. [20]).

In the sequel we will need the following:

**Definition 4.2.** Let  $G$  be a homeomorphism or diffeomorphism group of a manifold  $M$ .  $G$  is said to satisfy *Condition* (II) if for any disjoint, locally finite family of open relatively compact sets  $\{U_i\}$  of  $M$  and any family  $\{f_i\}$  of elements of  $G$  such that  $\text{supp}(f_i) \subset U_i$  one has  $\prod f_i \in G$ .

In order to show that  $BG$  is also acyclic it remains to have

**Proposition 4.3.** *Suppose that  $G$ , a subgroup of the group  $\text{Diff}^r(X, \text{rel } Z \times \{0\})$ , satisfies the deformation principle and Condition (II). Then the inclusion  $B_0G \rightarrow BG$  is a homotopy equivalence.*

**Proof.** (See also [20].) Let  $E_0G$  be the subspace of  $EG$  consisting of all simplices  $g_0 * g_1 * \dots * g_k$  such that for some sequence  $S \in \mathcal{S}$  as above all the diffeomorphisms  $g_0, \dots, g_k$  coincide in a neighborhood of  $Z \times S$ . Observe that  $E_0G$  is  $G$ -invariant, and  $E_0G/G = B_0G$ . Therefore, in view of Theorem 2.1 and properties of classifying spaces, the proof amounts to showing that  $E_0G$  is contractible.

It suffices to show that if  $\sigma_i = g_{i0} * \dots * g_{ip}$ ,  $i = 1, \dots, q$ , are a finite number of  $p$ -simplices of  $E_0G$  then there exists  $g \in G$  such that  $\sigma_i * g$  is contained in  $E_0G$  for all  $i$ . Suppose that  $\sigma_i$  is associated with  $S_i \in \mathcal{S}$ . Then we have to find another sequence  $S$  such that  $S \cap S_i$  is infinite for each  $i$ , and a diffeomorphism  $g \in G$  which coincides with  $g_{i0}$  in a neighborhood of  $Z \times (S_i \cap S)$  for each  $i$ . This will be done by induction.

The elements of the sequence  $S = \{s_n\}$  will be chosen increasingly so that  $s_n \in S_{\bar{n}}$  if and only if  $n = \bar{n} \pmod{q}$ . The first step: if  $\bar{s}_1$  is the first element of  $S_1$  we put  $s_1 = \bar{s}_1$  and choose a relative compact neighborhood  $U_1$  of  $Z \times \{s_1\}$ . Due to the deformation principle there exists  $g_1 \in G$  with  $\text{supp}(g_1) \subset U_1$  and such that  $g = g_{10}$  in a neighborhood of  $Z \times \{s_1\}$ . Next suppose that we have already chosen  $s_1, \dots, s_n \in S$ , relatively compact sets  $U_1, \dots, U_n$  such that  $U_j$  is a neighborhood of  $Z \times \{s_j\}$  and  $\bar{U}_j \cap \bar{U}_k = \emptyset$  if  $i \neq j$ , and  $g_1, \dots, g_n \in G$  with  $\text{supp}(g_j) \subset U_j$  and  $g_j = g_{0\bar{j}}$  in a neighborhood of  $Z \times \{s_j\}$ . Take  $s_{n+1}$  so large that there is a relatively compact neighborhood  $U_{n+1}$  of  $Z \times \{s_{n+1}\}$  satisfying  $\bar{U}_{n+1} \cap \bigcup_{i=1}^n \bar{U}_i = \emptyset$ . Now by the deformation principle there exists  $g_{n+1} \in G$  with  $\text{supp}(g_{n+1}) \subset U_{n+1}$  and  $g_{n+1} = g_{0\overline{n+1}}$  in a neighborhood of  $Z \times \{s_{n+1}\}$ . In view of Condition (II),  $g = \prod g_n \in G$ , and  $g$  is the required diffeomorphism.  $\square$

**5. The main result.** Since [3] it is well known that classical groups of diffeomorphisms satisfy the  $n$ -transitivity property. For our purpose this property is necessary in a different form.

According to the assumption on  $M$  from section 1, let  $Z = \partial\bar{M}$  and let  $X$  be a compact submanifold (with boundary) of  $M$  such that  $M \setminus X$  is homeomorphic to  $Z \times \mathbb{R}_+$  and the connected components of  $Z$  correspond to the ends of  $M$ .

**Definition 5.1.** A diffeomorphism group  $G(M) \subset \mathcal{H}(M)$  of a manifold  $M$  satisfies  *$n$ -transitivity at  $\infty$*  if the following conditions hold:

(i) For any two  $n$ -partitions of the interval  $[1, 2]$ :  $1 = t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} = 2$  and  $1 = t'_0 < t'_1 < t'_2 < \dots < t'_n < t'_{n+1} = 2$ , there is  $g \in G(M)$  such that  $\text{supp}(g) \subset Z \times (1, 2)$  and  $g(Z \times \{t_j\}) \subset Z \times \{t'_j\}$  for  $j = 0, 1, \dots, n+1$ .

(ii) For any compact subset  $B \subset Z \times (1, 2)$  and  $1 < t < 2$  there exist  $g, h \in G(M)$  such that  $\text{supp}(g), \text{supp}(h) \subset Z \times (1, 2)$ , and  $g(B) \subset Z \times (1, t)$ ,  $h(B) \subset Z \times (t, 2)$ .

(iii) For any reals  $0 < \alpha < \beta$  there exists  $g \in G_c(M)$  such that  $g(Z \times (1, 2)) = Z \times (\alpha, \beta)$ .

This property, as it is formulated, is often too strong and it could spoil proofs for particular geometric structures. In fact, it can be relaxed in many cases but we will not discuss this problem here.

**Lemma 5.2.** *If a subgroup  $G$  of  $\text{Diff}^r(X, \text{rel } Z \times \{0\})$  (where all the groups are endowed with the discrete topology) satisfies the deformation principle, the  $n$ -transitivity at  $\infty$  and Condition (II) then the assertion of Lemma 4.1 holds true for  $G$ .*

**Proof.** Every group  $G_S$  is isomorphic to an infinite product of copies of  $H := \{g \in G : \text{supp}(g) \subset Z \times (1, 2)\}$ . This isomorphism is uniquely determined by a choice of identifications, say  $g_i$ , of  $Z \times (1, 2)$  with subsequent sets  $Z \times (s_i, s_{i+1})$  in view of (iii). In fact, to any  $h \in G_S$  corresponds a unique sequence  $(h_i)$  of the group  $G$  with  $\text{supp}(h_i) \subset Z \times (s_i, s_{i+1})$  and with  $h_i = h$  on  $Z \times (s_i, s_{i+1})$  for all  $i$ . To show this we can use for each  $Z \times (s_i, s_{i+1})$  Corollary 3.5 as in the definition of  $\Sigma$  (before Lemma 4.1) combined with Condition (II). The required isomorphism is now determined by  $g_i$ .

Furthermore,  $H$  is a direct sum group (cf. [16], p. 308), i.e. there is a homomorphism  $H \times H \rightarrow H$ ,  $(h_1, h_2) \mapsto h_1 * h_2$  which verifies the following two conditions:

a) if  $h_1, \dots, h_m \in H$  then there are  $c, d \in H$  such that

$$c(h_j * \text{id})c^{-1} = d(\text{id} * h_j)d^{-1} = h_j$$

for all  $j$ ,



- b) for all three  $m$ -tuples  $h_1, \dots, h_m, h'_1, \dots, h'_m, h''_1, \dots, h''_m$  of elements of  $H$  there is  $c \in H$  such that

$$c(h_j * (h'_j * h''_j))c^{-1} = (h_j * h'_j) * h''_j.$$

This homomorphism is defined, up to conjugation, by using (ii) with  $t = \frac{3}{2}$ . The conditions a) and b) are fulfilled thanks to (ii) and (i), resp.

The product  $*$  can be regarded as the juxtaposition of diffeomorphisms in the  $\mathbb{R}$ -direction.

As above  $G_S$  is identified with  $H^S$ . On the other hand, we identify any  $(s \mapsto h|_{Z \times [0, s]}) \in G_c^S$  in the image of  $\Sigma$  with an element of  $H^S$  by using identifications of  $Z \times (1, 2)$  with the subsequent sets  $Z \times (0, s_i)$  in view of (iii). Then the homomorphism  $\Sigma$  identifies with the endomorphism of  $H^S$  (still denoted by  $\Sigma$ ) given by

$$(g_1, g_2, g_3, \dots) \mapsto (g_1, g_1 * g_2, g_1 * g_2 * g_3, \dots),$$

where  $g_1 * g_2 * \dots * g_n = g_1 * (g_2 * \dots * (g_{n-1} * g_n) \dots)$ . This identification can be done by using the condition (i) for all  $n$  and Condition (II), and  $\Sigma$  is defined up to conjugations.

Moreover, by using a componentwise argument together with Condition (II) it is apparent that also  $H^S$  carries a direct sum group structure. It follows from Proposition 2.3 that  $H_*(BH^S, \mathbb{R})$  admits a structure of a connected Hopf algebra with unit ([16]). Now, in view of [17], §1, there exists an inversion  $t : H_*(BH^S, \mathbb{R}) \rightarrow H_*(BH^S, \mathbb{R})$  such that  $t * \text{id} = \epsilon$ , where  $\epsilon$  is the unit in  $H_*(BH^S, \mathbb{R})$ . This enables to show that  $\Sigma$  is an equivalence as in [20].  $\square$

**Theorem 5.3.** *Let  $M = \text{Int } \bar{M}$ , where  $\bar{M}$  is a compact manifold with boundary. Let  $G(M) \subset \text{Diff}^r(M)_0$  be a connected locally contractible group of diffeomorphisms satisfying the following properties*

- (1)  $G_c(M)_0$  is perfect,
- (2)  $G(M)$  satisfies the deformation principle (Definition 3.1),
- (3)  $G(M)$  satisfies Condition (II) (Definition 4.2),
- (4)  $G(M)$  satisfies the  $n$ -transitivity at  $\infty$  for all  $n$  (Definition 5.1).

*Then  $G(M)$  is perfect. Moreover, assume that  $G(M)$  satisfies (2), (3), (4), and the following condition:*

- (5) *For any relatively compact ball  $B$  there is  $g \in G(M)$  such that  $\{g^i(B)\}_{i=0,1,\dots}$  (where  $g^0(B) = B$ ) is a disjoint, locally finite family.*

*Then  $G(M)$  is perfect as well.*

**Remark 5.4.** This theorem is not refined enough to tackle with some important groups. E.g. if a group is not perfect the present formulation is not adequate. An interesting and deep problem is to find relation between  $H_1(G)$  and  $H_1(G_c)$  for particular non-perfect groups (for instance, the symplectomorphism group).

**Proof.** Let  $Z = \partial\bar{M}$ . There is a compact submanifold (with boundary)  $X$  of  $M$  such that  $M \setminus X$  is homeomorphic to  $Z \times \mathbb{R}_+$  and the connected components of  $Z$  correspond to the ends of  $M$ . Let  $f \in G(M)$  be sufficiently close to the identity. By the deformation principle  $f$  can be decomposed as  $f = gh$ , where  $g, h \in G(M)$ ,  $g = \text{id}$  in a neighborhood of  $X$  ( $\text{supp}(g)$  need not be compact), and  $h \in G_c(M)_0$ . In light of the assumptions (2), (3), (4), Lemma 5.2, Lemma 4.1 and Proposition 2.2,  $g$  can be expressed as a product of commutators. The same is true for  $h$  due to (1) and the first assertion follows.

To prove the second assertion we may decompose the above  $h \in G_c(M)_0$  as  $h = h_1 \dots h_s$ , where each  $h_i$  is supported in a relatively compact ball (Corollary 3.3). Therefore we may and do assume that  $h$  is supported in a relatively compact ball  $B$ . Denote  $\bar{h} = \prod_{i=0}^{\infty} g^i h g^{-i}$ , and  $\tilde{h} = \prod_{i=1}^{\infty} g^i h g^{-i}$ , where  $g \in G(M)$  is as in the assumption (5). Then  $g^{-1} \tilde{h} g = \bar{h}$ , and  $h = (\tilde{h})^{-1} \bar{h}$ . Since a conjugation is the identity on the homology level [4], it follows that  $[h] = e$  in  $H_1(G(M))$ . This proves the second assertion.  $\square$

**6. Some examples.** There are several interesting open problems concerning the groups of homeomorphisms and diffeomorphisms with no restriction on support (some of them were mentioned on the list). Here we give simple examples which can be related to the presented method.

**Example 6.1.** Let  $M$  be a Lipschitz manifold and  $\mathcal{H}_{\mathcal{L}ip}(M)_0$  is the identity component (in a special Lipschitz topology) of the group of all (locally) Lipschitz homeomorphisms of  $M$ . Recently Abe and Fukui [1] proved that  $\mathcal{H}_{\mathcal{L}ip}(M)$  is perfect provided  $M$  is compact. One can use Theorem 5.3 to show that  $\mathcal{H}_{\mathcal{L}ip}(M)_0$  is still perfect for  $M$  open but the proof requires several preparatory results.

**Example 6.2.** Let  $n = \dim M$  and  $G = \text{Diff}^{n+1}(M)_0$ . It is still not known whether  $H_1(G_c) = 0$ . (Mather in [13] showed some analogous results which are true for  $r \neq n + 1$  and false for  $r = n + 1$ , but there is still no definite answer to the problem.) But for  $M = \mathbb{R}^n$  it is easy to see that  $H_1(G) = 0$ . Indeed, it follows directly from the second assertion of Theorem 5.3 by using a translation.

**Example 6.3.** Let  $G = \text{Diff}^\infty(\mathbb{R}^n, 0)$  be the group of orientation preserving diffeomorphisms fixing 0. By using Takens' normalization theorem Fukui [8] proved that if  $\Phi : G_c \rightarrow Gl^+(n, \mathbb{R})$  is the homomorphism which to  $\phi \in G_c$  assigns its Jacobi matrix at 0 then the equality

$$(6.1) \quad \ker \Phi = [\ker \Phi, G_c]$$

holds. We can extend  $\Phi$  to  $\tilde{\Phi} : G \rightarrow Gl^+(n, \mathbb{R})$  in the obvious way. Let  $B$  be the unit ball at 0 in  $\mathbb{R}^n$ . Any  $f \in G$  can be written as  $f = gh$ , where

$g = \text{id}$  in a neighborhood of  $B$ , and  $h \in G_c$ . Notice that the equality (6.1) takes the form

$$(6.2) \quad \ker \tilde{\Phi} = [\ker \tilde{\Phi}, G]$$

for the group  $G$ . In fact, if  $f \in \ker \tilde{\Phi}$  is written as  $f = gh$ , where  $g = \text{id}$  in a neighborhood of  $B$ , and  $h \in \ker \tilde{\Phi}$ , then in view of Lemma 5.2 we have  $g \in [\ker \tilde{\Phi}, \ker \tilde{\Phi}]$ , and due to (6.1) we obtain  $h \in [\ker \tilde{\Phi}, G_c]$ . Therefore we showed  $(\subset)$  of (6.2), and the converse inclusion is trivial. Now by the same argument as in [8] we have that  $H_1(G) = \mathbb{R}$ .

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