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Types of conditional convergence

Abstract. The aim of this paper is to investigate relations between different types of conditional convergence. Results presented in this paper generalize theorems obtained by P. Fernandez [2] and A. R. Padmanabhan [5].

1. Introduction. Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\mathcal{F}$ be a sub-σ-field contained in $\mathcal{A}$. We denote by $E^{\mathcal{F}}X$ conditional expectation of X with respect to $\mathcal{F}$. Let L^+ ($L^+(\mathcal{F})$) be the set of nonnegative random variables ($\mathcal{F}$-measurable). For $X \in L^+$ we can define the conditional expectation $E^{\mathcal{F}}X$ as in [4]. For $X = X^+ - X^-$, where $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$, we define the conditional expectation $E^{\mathcal{F}}X = E^{\mathcal{F}}X^+ - E^{\mathcal{F}}X^-$ if $\min(E^{\mathcal{F}}X^+, E^{\mathcal{F}}X^-) < \infty$ a.s. If $\max(E^{\mathcal{F}}X^+, E^{\mathcal{F}}X^-) < \infty$ a.s., then we say that $X \in L^1(\mathcal{F})$.

We will denote by $F_X(x) = P\{\omega\colon X(\omega) < x\}$ the distribution function for X, by ζ_{F_X} the set of continuity points of $F_X(x)$, that is $\zeta_{F_X} = \{x\colon F_X(x) = F_X(x+)\}$.

Definition 1.1. The conditional distribution function for random variable X given σ-field $\mathcal{F}$ we will call a process $F(x, \omega) = E^{\mathcal{F}} I_{[X<x]}(\omega)$ such that $F(x, \omega)$ is left-continuous and nondecreasing.

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Obviously $\lim_{x\to+\infty} F(x,\cdot) = 1$ a.s. and $\lim_{x\to-\infty} F(x,\cdot) = 0$ a.s. Note that if $x \in \zeta_{F_X}$, then $\lim_{\varepsilon\to 0} F(x-\varepsilon,\cdot) = F(x,\cdot)$ a.s.

We will say that random variables X and Y have the same conditional distribution if for each $x \in \mathbb{R}$

$$E^{\mathcal{F}} I_{[X<x]} = E^{\mathcal{F}} I_{[Y<x]} \text{ a.s.}$$

Note that if random variables X and Y have the same conditional distribution, then these variables have the same distribution, because for each $x \in \mathbb{R}$ we have

$$F_X(x) = P(X < x) = EI_{[X<x]} = E(E^{\mathcal{F}} I_{[X<x]}) = E(E^{\mathcal{F}} I_{[Y<x]}) = F_Y(x).$$

The following example shows that the opposite implication is not true.

Example 1. Let $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{B}([0,1]), \mu)$, where μ denotes Lebesgue measure, $\mathcal{F} = \mathcal{A}$,

$$X(\omega) = \begin{cases} 0, & \omega \in [0, \frac{1}{2}), \\ 1, & \omega \in [\frac{1}{2}, 1], \end{cases}$$

and $Y(\omega) = 1 - X(\omega)$. Then $F_X(x) = F_Y(x)$, but

$$I_{[X<\frac{1}{2}]}(\omega) = \begin{cases} 1, & \omega \in [0, \frac{1}{2}), \\ 0, & \omega \in [\frac{1}{2}, 1], \end{cases}$$

and

$$I_{[Y<\frac{1}{2}]}(\omega) = \begin{cases} 0, & \omega \in [0, \frac{1}{2}), \\ 1, & \omega \in [\frac{1}{2}, 1]. \end{cases}$$

Then $I_{[X<x]} \neq I_{[Y<x]}$ for $\omega \in \Omega$ and $x \in [0,1]$, hence these variables have the same distribution, but do not have the same conditional distribution.

Definition 1.2. We say that a sequence $\{X_n, n \geq 1\}$ of r.v. $\mathcal{F}$-conditionally converges in distribution to the r.v. X if for each $x \in \zeta_{F_X}$

$$E^{\mathcal{F}} I_{[X_n<x]} \longrightarrow E^{\mathcal{F}} I_{[X<x]} \text{ a.s., } n \to \infty.$$

This fact will be denoted by $X_n \xrightarrow{\mathcal{F}-D} X$.

Note that if $X_n \xrightarrow{\mathcal{F}-D} X$, then $X_n \xrightarrow{D} X$, since for each $x \in \zeta_{F_X}$ we have

$$\begin{aligned} \lim_{n\to\infty} F_n(x) &= \lim_{n\to\infty} EI_{[X_n<x]} = \lim_{n\to\infty} E\left(E^{\mathcal{F}} I_{[X_n<x]}\right) = E \lim_{n\to\infty} E^{\mathcal{F}} I_{[X_n<x]} \\ &= EE^{\mathcal{F}} I_{[X<x]} = F(x). \end{aligned}$$

The following example shows that the opposite implication is false.

Example 2. Let $(\Omega, \mathcal{A}, P)$ be defined as in the previous example, $\mathcal{F} = \sigma([0, \frac{1}{3}), [\frac{1}{3}, \frac{2}{3}), [\frac{2}{3}, 1])$, for even values of n

$$X_n(\omega) = \begin{cases} \frac{1}{4}, & \omega \in [0, \frac{1}{3}), \\ \omega, & \omega \in [\frac{1}{3}, \frac{2}{3}), \\ \frac{2}{3}, & \omega \in [\frac{2}{3}, 1], \end{cases}$$

while for odd values of n

$$X_n(\omega) = \begin{cases} \frac{2}{3}, & \omega \in [0, \frac{1}{3}), \\ \omega, & \omega \in [\frac{1}{3}, \frac{2}{3}), \\ \frac{1}{4}, & \omega \in [\frac{2}{3}, 1]. \end{cases}$$

Then $X_n \xrightarrow{D} X$, however for $x = \frac{1}{3}$ we have $E^{\mathcal{F}} I_{[X_n < \frac{1}{3}]} = E^{\mathcal{F}} I_{A_n} = I_{A_n}$, where

$$A_n = \begin{cases} [0, \frac{1}{3}) & \text{for } n = 2k, \\ [\frac{2}{3}, 1] & \text{for } n = 2k+1, \ k \in \mathbb{N}. \end{cases}$$

Since $\lim_{n\to\infty} I_{A_n}$ does not exist, the sequence $\{X_n, n \geq 1\}$ is not conditionally convergent in distribution.

Definition 1.3. We say that a sequence $\{X_n, n \geq 1\}$ of r.v. $\mathcal{F}$-conditionally converges in probability to the r.v. X if for every $\varepsilon > 0$

$$E^{\mathcal{F}} I_{[|X_n - X| > \varepsilon]} \longrightarrow 0 \quad \text{a.s.}, \ n \to \infty,$$

which will be denoted by $X_n \xrightarrow{\mathcal{F}-P} X$.

It is easily seen that if $X_n \xrightarrow{\mathcal{F}-P} X$, then $X_n \xrightarrow{P} X$, because

$$\begin{aligned} \lim_{n\to\infty} P(|X_n - X| > \varepsilon) &= \lim_{n\to\infty} E I_{[|X_n - X| > \varepsilon]} = \lim_{n\to\infty} E\left(E^{\mathcal{F}} I_{[|X_n - X| > \varepsilon]}\right) \\ &= E\left(\lim_{n\to\infty} E^{\mathcal{F}} I_{[|X_n - X| > \varepsilon]}\right) = 0. \end{aligned}$$

Theorem 1.4. *If $X_n \xrightarrow{\mathcal{F}-P} X$, then for every $\mathcal{F}$-measurable r.v. $\eta > 0$ a.s. $E^{\mathcal{F}} I_{[|X_n - X| > \eta]} \longrightarrow 0$ a.s., $n \to \infty$.*

Proof. Choose $\delta > 0$. Then

$$E^{\mathcal{F}} I_{[|X_n - X| > \eta]} \leq E^{\mathcal{F}} I_{[|X_n - X| > \delta, \ \eta \geq \delta]} + E^{\mathcal{F}} I_{[|X_n - X| > \eta, \ \eta < \delta]} \text{ a.s.}$$

Therefore

$$\lim_{n\to\infty} E^{\mathcal{F}} I_{[|X_n - X| > \eta]} \leq \lim_{n\to\infty} E^{\mathcal{F}} I_{[|X_n - X| > \delta]} + E^{\mathcal{F}} I_{[\eta < \delta]} = E^{\mathcal{F}} I_{[\eta < \delta]} \longrightarrow 0 \text{ a.s.},$$

if $\delta \to 0$. Since δ is arbitrary, we have proved the result. □

Let $\mathcal{F}$ and $\mathcal{G}$ be sub-σ-fields contained in the σ-field $\mathcal{A}$ and $\mathcal{F} \subset \mathcal{G}$. In such a case if $X_n \stackrel{\mathcal{G}-D}{\longrightarrow} X$, then $X_n \stackrel{\mathcal{F}-D}{\longrightarrow} X$, as for each $x \in \zeta_{F_X}$ we have

$$\lim_{n\to\infty} E^{\mathcal{F}} I_{[X_n<x]} = \lim_{n\to\infty} E^{\mathcal{F}} E^{\mathcal{G}} I_{[X_n<x]} = E^{\mathcal{F}} \lim_{n\to\infty} E^{\mathcal{G}} I_{[X_n<x]}$$
$$= E^{\mathcal{F}} E^{\mathcal{G}} I_{[X<x]} = E^{\mathcal{F}} I_{[X<x]} \quad \text{a.s.}$$

Similarly, if $X_n \stackrel{\mathcal{G}-P}{\longrightarrow} X$, then $X_n \stackrel{\mathcal{F}-P}{\longrightarrow} X$. Indeed, for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} E^{\mathcal{F}} I_{[|X_n-X|>\varepsilon]} = \lim_{n\to\infty} E^{\mathcal{F}} E^{\mathcal{G}} I_{[|X_n-X|>\varepsilon]}$$
$$= E^{\mathcal{F}} \lim_{n\to\infty} E^{\mathcal{G}} I_{[|X_n-X|>\varepsilon]} = 0 \quad \text{a.s.}$$

The opposite implications are not true.

Example 3. Let $(\Omega, \mathcal{A}, P)$ be defined as in the previous examples, $\mathcal{G} = \sigma([0, \frac{1}{2}), [\frac{1}{2}, 1])$, $\mathcal{F} = \{\emptyset, \Omega\}$. If

$$X_n(\omega) = \begin{cases} 1, & \omega \in [0, \frac{1}{2}), \\ 0, & \omega \in [\frac{1}{2}, 1], \end{cases}$$

for even values of n and $X_n(\omega) = 1 - X_{n-1}(\omega)$, for odd values of n then

$$\lim_{n\to\infty} E^{\mathcal{F}} I_{[X_n<x]} = \lim_{n\to\infty} P[X_n < x] = P[X < x] = E^{\mathcal{F}} I_{[X<x]} \text{ for } x \in \zeta_{F_X}.$$

However $E^{\mathcal{G}} I_{[X_n<\frac{1}{2}]} = E^{\mathcal{G}} I_{A_n} = I_{A_n}$, where

$$A_n(\omega) = \begin{cases} [0, \frac{1}{2}) & \text{for } n = 2k+1, \\ [\frac{1}{2}, 1] & \text{for } n = 2k,\ k \in \mathbb{N} \end{cases}$$

and $\lim_{n\to\infty} I_{A_n}$ does not exist.

Example 4. Let $(\Omega, \mathcal{A}, P)$ be defined as in the previous example. Every integer can be written in the form $n = 2^k + s$, where $k = \max\{l : 2^l \le n\}$, $s = 0, 1, ..., 2^k - 1$. If $\mathcal{G} = \mathcal{B}$, $\mathcal{F} = \{\emptyset, \Omega\}$ and

$$X_{2^k+s}(\omega) = \begin{cases} 1, & \omega \in [\frac{s}{2^k}, \frac{s+1}{2^k}), \\ 0, & \text{otherwise}, \end{cases}$$

then for every $\varepsilon > 0$,

$$E^{\mathcal{F}} I_{[|X_n|>\varepsilon]} = P[|X_n| > \varepsilon] = \frac{1}{2^k} \longrightarrow 0, \quad n \to \infty.$$

However for every $\varepsilon > 0$, $E^{\mathcal{G}} I_{[|X_n|>\varepsilon]} = I_{[|X_n|>\varepsilon]}$. Thus for $\varepsilon = \frac{1}{2}$ we have $\limsup_{n\to\infty} I_{[|X_n|>\varepsilon]} = 1$ a.s. but $\liminf_{n\to\infty} I_{[|X_n|>\varepsilon]} = 0$ a.s. Hence it is not true that $X_n \stackrel{\mathcal{G}-P}{\longrightarrow} X$, but $X_n \stackrel{\mathcal{F}-P}{\longrightarrow} X$.

For $\mathcal{F} = \mathcal{A}$ convergence $X_n \stackrel{\mathcal{F}-P}{\longrightarrow} X$ implies a.s. convergence. Indeed, $E^{\mathcal{F}} I_{[|X_n - X| > \varepsilon]} = I_{[|X_n - X| > \varepsilon]}$ a.s. and

$$\limsup_{n\to\infty} I_{[|X_n - X|>\varepsilon]} = \lim_{n\to\infty} \sup_{k\geq n} I_{[|X_k - X|>\varepsilon]} = \lim_{n\to\infty} I_{\bigcup_{k=n}^{\infty}[|X_k - X|>\varepsilon]} = 0 \text{ a.s.}$$

Hence

$$\lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} [|X_k - X| > \varepsilon]\right) = 0,$$

which is equivalent to a.s. convergence. On the other hand, the last statement implies the previous one, as can be easily seen. Thus a continuous link between convergence in probability and a.s. convergence is established.

2. Main results.

Theorem 2.1. *If $X_n \stackrel{\mathcal{F}-P}{\longrightarrow} X$, then for each $x \in \zeta_{F_X}$*

$$E^{\mathcal{F}} \left| I_{[X_n < x]} - I_{[X<x]} \right| \longrightarrow 0 \quad a.s., \; n \to \infty.$$

Proof. If $X_n \stackrel{\mathcal{F}-P}{\longrightarrow} X$, then for every $\varepsilon > 0$, $\lim_{n\to\infty} E^{\mathcal{F}} I_{[|X_n - X| > \varepsilon]} = 0$ a.s.
Moreover

$$\begin{aligned}
E^{\mathcal{F}} \left| I_{[X_n<x]} - I_{[X<x]} \right| &= E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} \\
&= E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} I_{[|X_n - X|\geq\varepsilon]} + E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} I_{[|X_n-X|<\varepsilon]} \\
&= E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} I_{[|X_n - X|\geq\varepsilon]} + E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} I_{[X-\varepsilon<X_n<X+\varepsilon]} \\
&\leq E^{\mathcal{F}} I_{[|X_n-X|\geq\varepsilon]} + E^{\mathcal{F}} I_{([X_n<x]\setminus[X<x])} I_{[X-\varepsilon<X_n<X+\varepsilon]} \\
&\qquad\qquad + E^{\mathcal{F}} I_{([X<x]\setminus[X_n<x])} I_{[X-\varepsilon<X_n<X+\varepsilon]} \\
&\leq E^{\mathcal{F}} I_{[|X_n-X|\geq\varepsilon]} + E^{\mathcal{F}} I_{[x\leq X<x+\varepsilon]} + E^{\mathcal{F}} I_{[x-\varepsilon\leq X<x]} \\
&= E^{\mathcal{F}} I_{[|X_n-X|\geq\varepsilon]} + F_X^{\mathcal{F}}(x+\varepsilon) - F_X^{\mathcal{F}}(x) + F_X^{\mathcal{F}}(x) - F_X^{\mathcal{F}}(x-\varepsilon) \quad \text{a.s.}
\end{aligned}$$

Thus for every $\varepsilon > 0$

$$\limsup_{n\to\infty} E^{\mathcal{F}} \left| I_{[X_n<x]} - I_{[X<x]} \right| \leq F_X^{\mathcal{F}}(x+\varepsilon) - F_X^{\mathcal{F}}(x-\varepsilon) \quad \text{a.s.}$$

Since ε is arbitrary, we have $\limsup_{n\to\infty} E^{\mathcal{F}} \left| I_{[X_n<x]} - I_{[X<x]} \right| = 0$ a.s. and hence $\lim_{n\to\infty} E^{\mathcal{F}} \left| I_{[X_n<x]} - I_{[X<x]} \right| = 0$ a.s. for $x \in \zeta_{F_X}$. □

Corollary 2.2. *If $X_n \stackrel{\mathcal{F}-P}{\longrightarrow} X$, then $X_n \stackrel{\mathcal{F}-D}{\longrightarrow} X$.*

Theorem 2.3. $\lim_{n\to\infty} E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} = 0$ *a.s.*, $x \in \zeta_{F_X}$ *if and only if for every* $D \in \mathcal{A}$ *and each* $x \in \zeta_{F_X}$

$$\lim_{n\to\infty} E^{\mathcal{F}} I_{[X_n<x]} I_D = E^{\mathcal{F}} I_{[X<x]} I_D \quad a.s.$$

Proof. Necessity. If $\lim_{n\to\infty} E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} = 0$ a.s. then by the previous theorem for every $x \in \zeta_{F_X}$ we have

$$\begin{aligned} \left|E^{\mathcal{F}} I_{[X_n<x]} I_D - E^{\mathcal{F}} I_{[X<x]} I_D\right| &= \left|E^{\mathcal{F}}\left(I_{[X_n<x]} - I_{[X<x]}\right) I_D\right| \\ &\le E^{\mathcal{F}}\left|I_{[X_n<x]} - I_{[X<x]}\right| \\ &= E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} \longrightarrow 0 \quad \text{a.s., } n\to\infty. \end{aligned}$$

Sufficiency. If $D = [X < x]$ and $x \in \zeta_{F_X}$, then $\lim_{n\to\infty} E^{\mathcal{F}} I_{[X_n<x]} I_{[X<x]} = E^{\mathcal{F}} I_{[X<x]}$. Hence

$$\begin{aligned} &\lim_{n\to\infty} E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} \\ &\quad = \lim_{n\to\infty} E^{\mathcal{F}}(I_{[X_n<x]} - I_{[X_n<x][X<x]} + I_{[X<x]} - I_{[X_n<x][X<x]}) \\ &\quad = \lim_{n\to\infty} E^{\mathcal{F}} I_{[X_n<x]} - \lim_{n\to\infty} E^{\mathcal{F}} I_{[X_n<x][X<x]} \\ &\qquad\qquad + E^{\mathcal{F}} I_{[X<x]} - \lim_{n\to\infty} E^{\mathcal{F}} I_{[X_n<x][X<x]} = 0 \quad \text{a.s.} \end{aligned}$$

□

Note that $\lim_{n\to\infty} E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} = 0$ a.s. for $x \in \zeta_{F_X}$ is equivalent to $\lim_{n\to\infty} E^{\mathcal{F}} I_{([X_n\le x]\triangle[X\le x])} = 0$ a.s. for $x \in \zeta_{F_X}$.

Theorem 2.4. *If* $\lim_{n\to\infty} E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} = 0$ *a.s. for each* $x \in \zeta_{F_X}$, *then* $X_n \overset{\mathcal{F}-P}{\longrightarrow} X$.

Proof. Let $B(x, r) = \{y\colon |x-y| < r\}$ and $\varepsilon > 0$. Let $\{x_i\}_{i=1,2\dots}$ be a countable dense subset of $\mathbb{R}$. Select γ such that $0 < \gamma < \frac{\varepsilon}{2}$ and $P[|X-x_i| = \gamma] = 0$. Then $E^{\mathcal{F}} I_{[|X-x_i|=\gamma, i=1,2,\dots]} = 0$ a.s. It is clear that $\bigcup_{i=1}^{\infty} B(x_i, \gamma) = \mathbb{R}$, therefore by continuity of measure P, we have $\lim_{t\to\infty} P\left[X \in \bigcup_{s=1}^{t} B(x_s, \gamma)\right] = 1$, hence

$$\lim_{t\to\infty} E^{\mathcal{F}} I_{\left[X\in\bigcup_{s=1}^{t} B(x_s,\gamma)\right]} = 1 \quad \text{a.s.}$$

For $0 < \delta < 1$ we define a random variable

$$N(\omega) = \inf\left\{t\colon E^{\mathcal{F}} I_{\left[X\in\bigcup_{s=1}^{t} B(x_s,\gamma)\right]} > 1-\delta\right\}.$$

Note that $A_n = [N(\omega) = n] \in \mathcal{F}$ and $P\left[\bigcup_{n=1}^{\infty} A_n\right] = 1$, i.e. $N < \infty$ a.s.

Moreover, if $K_t = \bigcup_{s=1}^{t} B(x_s,\gamma)$, then

$$\begin{aligned}
E^{\mathcal{F}} I_{[|X_k - X|>\varepsilon]} &= E^{\mathcal{F}} I_{[|X_k - X|>\varepsilon, \bigcup_{n=1}^{\infty} A_n]} = \sum_{n=1}^{\infty} E^{\mathcal{F}} I_{[|X_k - X|>\varepsilon]} I_{A_n} \\
&= \sum_{n=1}^{\infty} E^{\mathcal{F}} I_{[X\notin K_N, |X_k - X|>\varepsilon]} I_{A_n} + \sum_{n=1}^{\infty} E^{\mathcal{F}} I_{[X\in K_N, |X_k - X|>\varepsilon]} I_{A_n} \\
&\le \sum_{n=1}^{\infty} E^{\mathcal{F}} I_{[X\notin K_N]} I_{A_n} + \sum_{n=1}^{\infty} E^{\mathcal{F}} I_{[X\in K_N, |X_k - X|>\varepsilon]} I_{A_n} \\
&\le \delta + \sum_{n=1}^{\infty} E^{\mathcal{F}} I_{\left(\bigcup_{s=1}^{N} [X\in B(x_s,\gamma)]\cap[|X_k - X|>\varepsilon]\right)} I_{A_n} \ .
\end{aligned}$$

Thus, since

$$\begin{aligned}
&\bigcup_{s=1}^{N} [X\in B(x_s,\gamma)] \cap [|X_k - X| > \varepsilon] \\
&\quad \subset \bigcup_{s=1}^{N} [x_s - \gamma < X < x_s + \gamma] \cap [|X_k - X| > \varepsilon] \\
&\quad \subset \bigcup_{s=1}^{N} \left([x_s - \gamma < X < x_s + \gamma] \cap [x_s - \gamma < X_k < x_s + \gamma]^C\right) \\
&\quad \subset \left(\bigcup_{s=1}^{N} \left([x_s - \gamma < X < x_s + \gamma] \cap [X_k < x_s + \gamma]^C\right)\right) \\
&\qquad\qquad \cup \left(\bigcup_{s=1}^{N} \left([x_s - \gamma < X < x_s + \gamma] \cap [x_s - \gamma < X_k]^C\right)\right) \\
&\quad \subset \left(\bigcup_{s=1}^{N} \left([X < x_s + \gamma] \cap [X_k < x_s + \gamma]^C\right)\right) \\
&\qquad\qquad \cup \left(\bigcup_{s=1}^{N} \left([x_s - \gamma < X] \cap [x_s - \gamma < X_k]^C\right)\right) \\
&\quad \subset \left(\bigcup_{s=1}^{N} \left([X < x_s + \gamma] \cap [X_k < x_s + \gamma]^C\right)\right) \\
&\qquad\qquad \cup \left(\bigcup_{s=1}^{N} \left([X_k \le x_s - \gamma] \cap [X \le x_s - \gamma]^C\right)\right),
\end{aligned}$$

we have

$$\begin{aligned} &E^{\mathcal{F}} I_{[|X_k - X| > \varepsilon]} \\ &\quad \le \delta + \sum_{n=1}^{\infty} E^{\mathcal{F}} \left(\sum_{s=1}^{N} I_{([X < x_s + \gamma] \triangle [X_k < x_s + \gamma])} I_{A_n} \right) \\ &\qquad\qquad + \sum_{n=1}^{\infty} E^{\mathcal{F}} \left(\sum_{s=1}^{N} I_{([X_k \le x_s - \gamma] \triangle [X \le x_s - \gamma])} I_{A_n} \right) \quad \text{a.s.} \end{aligned}$$

Hence

$$\begin{aligned} &\limsup_{k\to\infty} E^{\mathcal{F}} I_{[|X_k - X| > \varepsilon]} \\ &\quad \le \delta + \lim_{k\to\infty} \sum_{n=1}^{\infty} E^{\mathcal{F}} \left(\sum_{s=1}^{N} I_{([X < x_s + \gamma] \triangle [X_k < x_s + \gamma])} I_{A_n} \right. \\ &\qquad\qquad \left. + \sum_{s=1}^{N} I_{([X_k \le x_s - \gamma] \triangle [X \le x_s - \gamma])} I_{A_n} \right) \\ &\quad = \delta + \sum_{n=1}^{\infty} \sum_{s=1}^{n} \lim_{k\to\infty} I_{A_n} E^{\mathcal{F}} I_{([X < x_s + \gamma] \triangle [X_k < x_s + \gamma])} \\ &\qquad\qquad + \sum_{n=1}^{\infty} \sum_{s=1}^{n} \lim_{k\to\infty} I_{A_n} E^{\mathcal{F}} I_{([X_k \le x_s - \gamma] \triangle [X \le x_s - \gamma])} = \delta \quad \text{a.s.} \end{aligned}$$

Since δ is arbitrary, we have proved the result. □

Theorem 2.5. *Let X be $\mathcal{F}$-measurable random variable. If $X_n \xrightarrow{\mathcal{F}-D} X$, then $X_n \xrightarrow{\mathcal{F}-P} X$.*

Proof. If $X_n \xrightarrow{\mathcal{F}-D} X$, then for each $x \in \zeta_{F_X}$ we have $\lim_{n\to\infty} E^{\mathcal{F}} I_{[X_n < x]} = E^{\mathcal{F}} I_{[X<x]} = I_{[X<x]}$ a.s. and $\lim_{n\to\infty} E^{\mathcal{F}} I_{[X_n<x]} I_{[X<x]} = I_{[X<x]}$. Hence

$$\begin{aligned} \lim_{n\to\infty} E^{\mathcal{F}} I_{([X_n<x]\triangle[X<x])} &= \lim_{n\to\infty} E^{\mathcal{F}} (I_{[X_n<x]} - I_{[X_n<x]} I_{[X<x]}) \\ &\quad + \lim_{n\to\infty} E^{\mathcal{F}} (I_{[X<x]} - I_{[X_n<x]} I_{[X<x]}) = 0 \quad \text{a.s.} \end{aligned}$$

Therefore by the previous theorem we have that $X_n \xrightarrow{\mathcal{F}-P} X$. □

Note that if $\mathcal{F}$ is the trivial σ-field ($\mathcal{F} = \{\emptyset, \Omega\}$), then we obtain the following well-known result.

Corollary 2.6. *Let C be a random variable such that $P[C = c] = 1$ for some $c \in \mathbb{R}$. Then $X_n \xrightarrow{D} C$ if and only if $X_n \xrightarrow{P} C$.*

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