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Isoptics of open rosettes

Abstract. In this paper we introduce the notion of an open rosette and its isoptics and prove some theorems such as the sine theorem and its inverse in this framework.

Definition 1. A plane open curve C with positive curvature is said to be an open rosette.

Let $C : z = z(t)$, $t \in \mathbb{R}$ be an open rosette and choose a coordinate system with point O as its center. We choose such $t_0 \in \mathbb{R}$ that tangent line to C in $z(t_0)$ is perpendicular to Ox-axis. If there are many such points we can choose any of them. Next, at a point $z(t)$ we define vector $e^{it} = \cos t + i \sin t$ (as in Figure 1) where t is oriented angle between the positive direction of Ox-axis and e^{it}. Now we are going to define an oriented support function $p(t)$ of curve C. If e^{it} points to a half plane which doesn't contain the origin of the coordinate system we define $p(t)$ as ordinary distance between O and tangent line at $z(t)$. If not we define $p(t)$ as minus ordinary distance.

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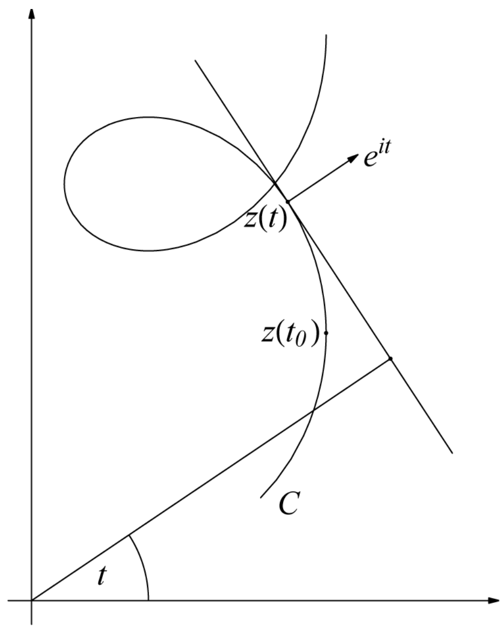

FIGURE 1. A support function of an open rosette.

Theorem 1. *Let C be an open rosette. Then the rosette C can be parametrized by*

$$z(t) = p(t)e^{it} + p'(t)ie^{it}$$

where $t \in (a, b)$, $a, b \in \mathbb{R}$, and $p(t)$ is its support function.

Proof of this fact is well known and can be found in [7].

Remark 1. Curvature of an open rosette parametrized by the support function is given by

$$k(t) = \frac{1}{p(t) + p''(t)}, \quad t \in (a, b)$$

Since we assume that rosette has positive curvature, the function $p(t)$ is at least of class C^2.

Example 1. Consider curve with the following support function

$$p(t) = \frac{1}{\sin t} \quad \text{for } t \in (0, \pi).$$

Then

$$p''(t) = \frac{\sin^2 t + 2 \sin t \cos^2 t}{\sin^4 t}.$$

$k(t) > 0$ for $t \in (0, \pi)$ so it is an open rosette.

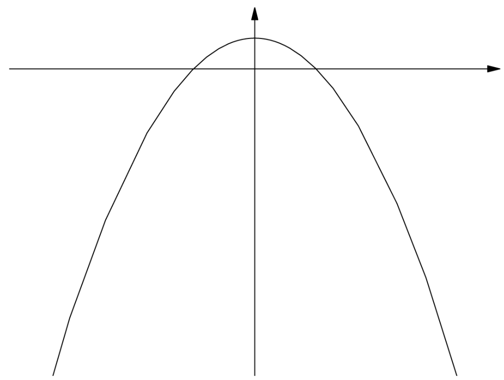

FIGURE 2. Example 1.

Example 2. Consider another function

$$p(t) = t^2 \quad \text{for } t \in \mathbb{R}.$$

Then

$$p''(t) = 2$$

and we get an open rosette.

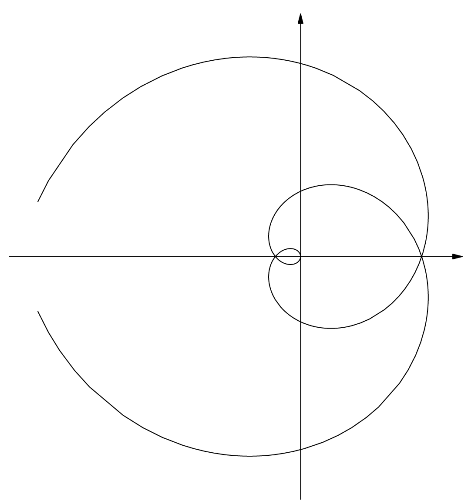

FIGURE 3. Example 2.

Remark 2. In the further part of this paper we consider only open rosettes for which $t \in \mathbb{R}$.

Let C be on open rosette with a support function $p(t)$, $t \in \mathbb{R}$. We fix a point $z(t_0)$ and denote the tangent line to C at $z(t_0)$ by l_0. Next, we choose

t_1, $t_0 < t_1$ closest to t_0 in the sense of parametrization and we denote the tangent line at this point by l_1. We choose t_1 in such a way that an angle between tangents l_0 and l_1 equals $\pi - \alpha$ (see Figure 4).

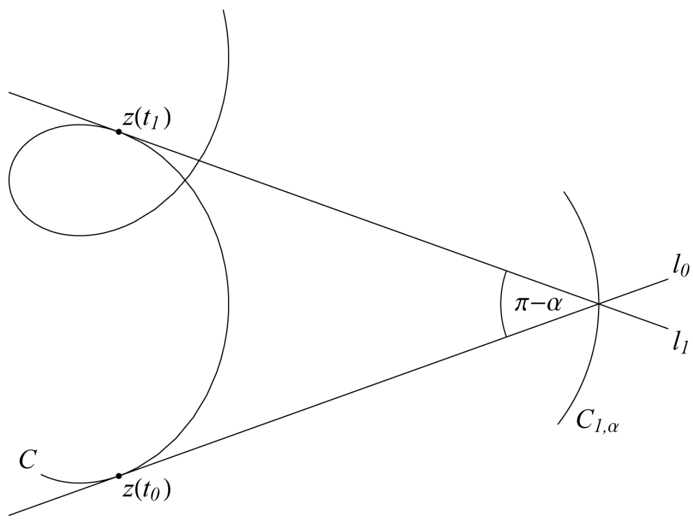

FIGURE 4. α-isoptic of open rosette.

Definition 2. The cut locus of common points of the tangents l_0 and l_1 forms α-isoptic of the 1-st order of an open rosette. We denote it by $C_{1,\alpha}$.

Remark 3. In the same manner as in [5] and [1] we show that

$$t_1 = t_0 + \alpha$$

and

$$z_{\alpha,1}(t) = p(t)e^{it} + \left(-p(t)\cot\alpha + \frac{p(t+\alpha)}{\sin\alpha}\right) ie^{it}.$$

We define the following points

$$\begin{gathered} t_2 = t_1 + 2\pi = t_0 + \alpha + 2\pi \\ t_3 = t_2 + 2\pi = t_0 + \alpha + 4\pi \\ \dots\dots\dots\dots\dots\dots \\ t_k = t_{k-1} + 2\pi = t_0 + \alpha + 2(k-1)\pi \\ \dots\dots\dots\dots\dots\dots \end{gathered}$$

We denote by $l_1, l_2, \dots, l_k, \dots$ (see Figure 5) tangent lines to C at these points.

Definition 3. The cut locus of common points of the tangents l_0 and l_k forms α-isoptic of k-th order of an open rosette. We denote this curve by $C_{k,\alpha}$ and call it k,α-isoptic.

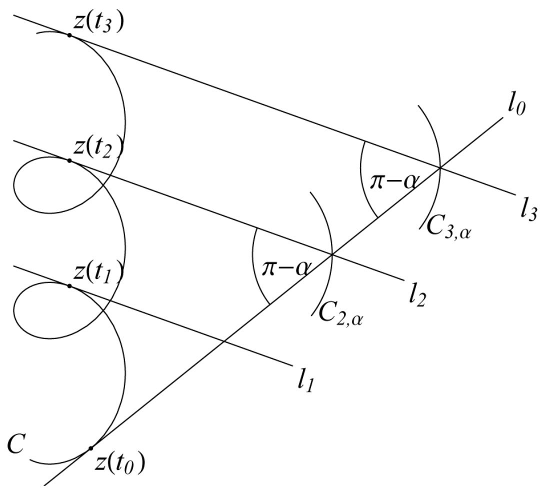

FIGURE 5. k, α-isoptics of an open rosette.

Remark 4. Let C be an open rosette. Since $t \in \mathbb{R}$ then $t_k = t_0 + \alpha + 2(k-1)\pi \in \mathbb{R}$, so there exists k, α-isoptic for any k.

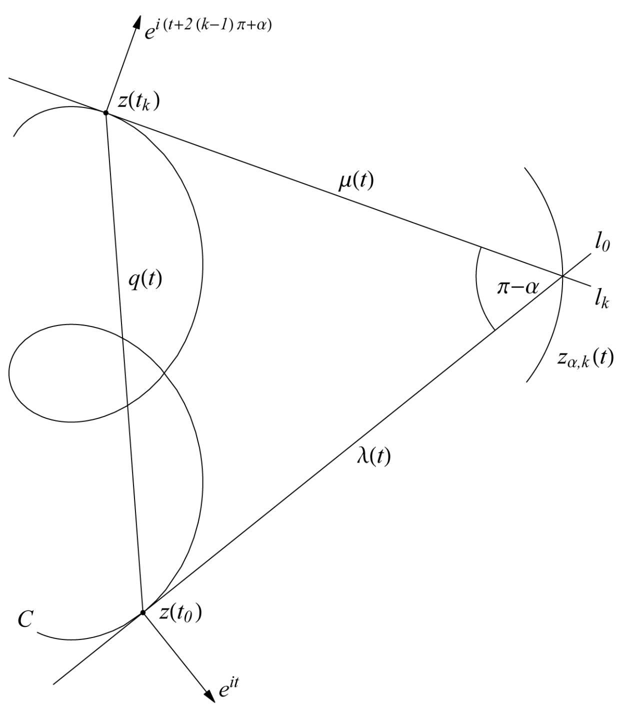

FIGURE 6. Parametrization of an k, α-isoptic.

Theorem 2 (Parametrization of k,α-isoptic)**.** *For α-isoptic of k-th order of an open rosette we have the following equation*

$$z_{\alpha,k}(t) = p(t)e^{it} + \left(-p(t)\cot\alpha + \frac{p(t+2(k-1)\pi+\alpha)}{\sin\alpha}\right) ie^{it}$$

for $t \in \mathbb{R}, k \in \mathbb{N}$.

Proof. For $u = a+bi$, $w = c+di$ we denote $[u,w] = ad-bc$. Then we have

$$\begin{aligned} q(t) &= z(t) - z(t_k) \\ &= p(t)\cos t - p'(t)\sin t - p(t+2(k-1)\pi+\alpha)\cos(t+\alpha) \\ &\quad + p'(t+2(k-1)\pi+\alpha)\sin(t+\alpha) \\ &\quad + \big(p(t)\sin t + p'(t)\cos t - p(t+2(k-1)\pi+\alpha)\sin(t+\alpha) \\ &\qquad - p'(t+2(k-1)\pi+\alpha)\cos(t+\alpha)\big)\, i, \end{aligned}$$

$$b(t) = \left[q(t), e^{it}\right] = p(t+2(k-1)\pi+\alpha)\sin\alpha + p'(t+2(k-1)\pi+\alpha)\cos\alpha - p'(t),$$

$$B(t) = \left[q(t), ie^{it}\right] = p(t) - p(t+2(k-1)\pi+\alpha)\cos\alpha + p'(t+2(k-1)\pi+\alpha)\sin\alpha.$$

We can write a parametrization of a k,α-isoptic as

$$z_{\alpha,k}(t) = z(t) + \lambda(t)ie^{it}$$

or

$$z_{\alpha,k}(t) = z(t+2(k-1)\pi+\alpha) + \mu(t)ie^{i(t+2(k-1)\pi+\alpha)},$$

where $\lambda(t)$ and $\mu(t)$ are functions of the appropriate class. Thus, we have

$$z(t) + \lambda(t)ie^{it} = z(t+2(k-1)\pi+\alpha) + \mu(t)ie^{i(t+2(k-1)\pi+\alpha)}$$

and hence

$$z(t) - z(t+2(k-1)\pi+\alpha) = \mu(t)ie^{i(t+2(k-1)\pi+\alpha)} - \lambda(t)ie^{it}.$$

Multiplying the above equations by e^{it} and ie^{it} we obtain

$$\left[z(t) - z(t+2(k-1)\pi+\alpha), e^{it}\right] = \left[\mu(t)ie^{i(t+2(k-1)\pi+\alpha)} - \lambda(t)ie^{it}, e^{it}\right]$$

$$\left[z(t) - z(t+2(k-1)\pi+\alpha), ie^{it}\right] = \left[\mu(t)ie^{i(t+2(k-1)\pi+\alpha)} - \lambda(t)ie^{it}, ie^{it}\right].$$

Making use of the fact that left-hand sides are equal $b(t)$ and $B(t)$ we obtain

$$\lambda(t) - \mu(t)\cos\alpha = b(t)$$

$$-\mu(t)\sin\alpha = B(t)$$

which gives

$$\lambda(t) = b(t) - B(t)\cot\alpha$$

$$\mu(t) = \frac{-B(t)}{\sin\alpha}.$$

Combining this we obtain

$$\begin{aligned} z_{\alpha,k}(t) &= z(t) + \lambda(t) i e^{i(t)} \\ &= p(t)e^{it} + \left(-p(t)\cot\alpha + \frac{p(t+2(k-1)\pi+\alpha)}{\sin\alpha} \right) i e^{it}. \end{aligned}$$

□

Example 3. Let us consider the following function

$$p(t) = t^2 \quad \text{for } t \in \mathbb{R}.$$

We have the following parametrization of its k, α-isoptics

$$z_{\alpha,k}(t) = t^2 e^{it} + \left(-t^2 \cot\alpha + \frac{(t+2(k-1)\pi+\alpha)^2}{\sin\alpha} \right) i e^{it} \quad \text{for } t \in \mathbb{R}.$$

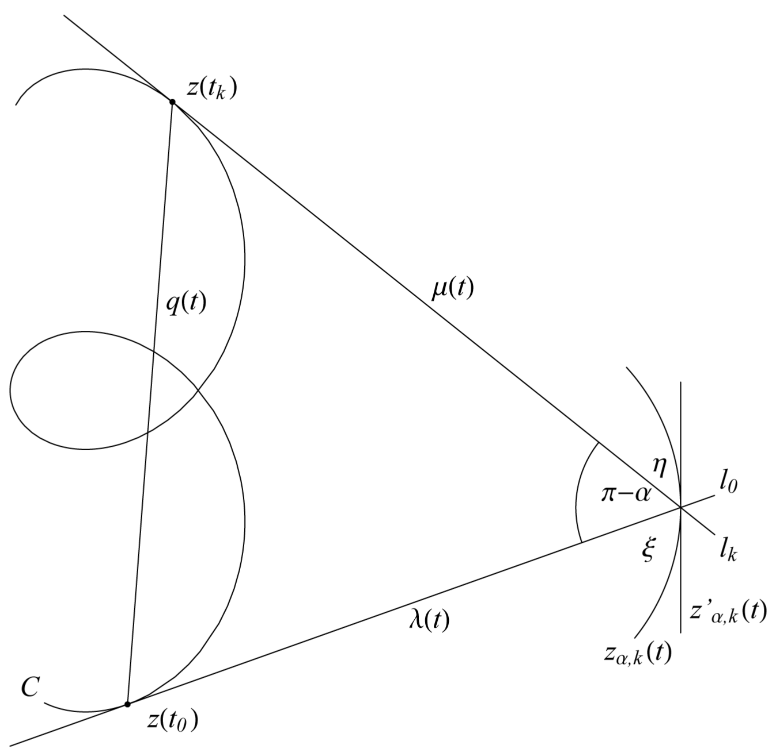

Figure 7. Sine theorem.

Theorem 3 (Sine theorem)**.** *Under the notations of Figure 7 for an open rosette C and its α-isoptic of k-th order the following equalities hold*

$$\frac{|q(t)|}{\sin\alpha} = \frac{\lambda(t)}{\sin\xi} = \frac{\mu(t)}{\sin\eta}.$$

Proof. We have

$$b'(t) = R(t + 2(k-1)\pi + \alpha)\cos\alpha - R(t) + B(t),$$
$$B'(t) = R(t + 2(k-1)\pi + \alpha)\sin\alpha - b(t),$$

where $R(t) = p(t) + p''(t)$ is the curvature radius of curve C. Next, we obtain

$$z'_\alpha(t) = \varrho(t) i e^{it} - \lambda(t) e^{it},$$

where

$$\varrho(t) = B(t) + b(t)\cot\alpha.$$

It is obvious that

$$\frac{1}{2}\left[ie^{it}, z'_\alpha(t)\right] = \frac{1}{2}\left|ie^{it}\right|\left|z'_\alpha(t)\right|\sin\xi$$

so

$$\frac{\left[ie^{it}, z'_\alpha(t)\right]}{\sin\xi} = \left|z'_\alpha(t)\right|.$$

We have

$$\left[ie^{it}, z'_\alpha(t)\right] = \lambda(t)$$

and

$$\left|z'_\alpha(t)\right| = \frac{\sqrt{b^2(t) + B^2(t)}}{\sin\alpha}.$$

It is a simple matter to check that

$$|q(t)| = \frac{\sqrt{b^2(t) + B^2(t)}}{\sin\alpha}$$

so

$$\left|z'_\alpha(t)\right| = \frac{|q(t)|}{\sin\alpha}$$

thus

$$\frac{|q(t)|}{\sin\alpha} = \frac{\lambda(t)}{\sin\xi}.$$

In the same manner we can see that

$$\frac{1}{2}\left[ie^{i(t+2(k-1)\pi+\alpha)}, z'_\alpha(t)\right] = \frac{1}{2}\left|ie^{i(t+2(k-1)\pi+\alpha)}\right|\left|z'_\alpha(t)\right|\sin\eta$$

so

$$\frac{\left[ie^{i(t+2(k-1)\pi+\alpha)}, z'_\alpha(t)\right]}{\sin\eta} = \left|z'_\alpha(t)\right|.$$

Hence

$$\left[ie^{i(t+2(k-1)\pi+\alpha)}, z'_\alpha(t)\right] = \mu(t)$$

and we obtain

$$\frac{\mu(t)}{\sin\eta} = \frac{|q(t)|}{\sin\alpha}.$$

Summarizing we have

$$\frac{|q(t)|}{\sin\alpha} = \frac{\lambda(t)}{\sin\xi} = \frac{\mu(t)}{\sin\eta}.$$

□

We now consider the inverse problem. We will prove the following:

Theorem 4. *Let $C: z(t) = p(t)e^{it} + p'(t)ie^{it}$ be a given open rosette and γ be an open curve. Assume that there exists differentiable function $\phi: \mathbb{R} \to \mathbb{R}$ such that, the tangent lines to C at $z(t)$ and $z(\phi(t))$ intersect for each t at some point of γ. We assume that at these points the sine formula holds. Then γ is an isoptic of C for some k and α.*

Proof. The function $\phi(t)$ defines the function $\alpha(t)$, $0 < \alpha(t) < \pi$, which represents the oriented angle between the tangent lines to C at $z(t)$ and $z(\phi(t))$. It is easily seen that $\phi(t) = t + 2(k-1)\pi + \alpha(t)$ for some uniquely determined integer k.

Analogously to the proof of Theorem 2 we obtain parametrization of curve γ:

$$z_\gamma(t) = p(t)e^{it} + \left(-p(t)\cot\alpha(t) + \frac{1}{\sin\alpha(t)}p(t + 2(k-1)\pi + \alpha(t))\right)ie^{it}$$

and values of

$$\lambda_\gamma(t) = -p(t)\cot\alpha(t) - p'(t) + \frac{p(t + 2(k-1)\pi + \alpha(t))}{\sin\alpha(t)}$$

$$\mu_\gamma(t) = p(t + 2(k-1)\pi + \alpha(t))\cot\alpha(t) - \frac{p(t)}{\sin\alpha(t)} - p'(t + 2(k-1)\pi + \alpha(t)).$$

Under the assumptions of theorem we have

$$\frac{\lambda_\gamma(t)}{\sin\xi} = \frac{\mu_\gamma(t)}{\sin\eta}.$$

The above condition is equivalent to

$$\frac{\lambda_\gamma(t)}{\left[ie^{it}, z'_\gamma(t)\right]} = \frac{\mu_\gamma(t)}{\left[ie^{i(t+2(k-1)\pi+\alpha(t))}, z'_\gamma(t)\right]}.$$

From this formula it follows that

$$\left[ie^{it}, z'_\gamma(t)\right] = \lambda_\gamma(t)$$

and

$$\left[ie^{i(t+2(k-1)\pi+\alpha(t))}, z'_\gamma(t)\right] = \mu_\gamma(t)(1 + \alpha'(t)).$$

Thus we obtain

$$\frac{\lambda_\gamma(t)}{\lambda_\gamma(t)} = \frac{\mu_\gamma(t)}{\mu_\gamma(t)(1 + \alpha'(t))}$$

so

$$\alpha'(t) = 0$$

which means that $\alpha(t) = \text{const}$. □

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