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On estimating the coefficient product $A_1A_2A_3$ for real bounded non-vanishing univalent functions

Abstract. The class of the title is sufficiently limited for allowing certain estimations for combinations of the three first coefficients A_1, A_2 and A_3. The negative sign of A_2 implies complications which, however, in the present treatment will be governed, when estimating the product $A_1A_2A_3$.

1. Introduction. In [2] the observations of J. Śladkowska [1] were utilized in determining the first coefficient bodies for functions F which are univalent and bounded with the condition of non-vanishedness. Denote the class of these functions by $S'(B)$. Another condition will be a restriction to real coefficients A_ν. The subclass thus introduced is denoted by $S'_R(B)$:

$$\begin{cases} S'(B) = \{F \mid F(z) = B + A_1 z + \dots,\ z \in U \supset F(U) \not\ni O, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < B < 1,\ A_1 > 0\}, \\ S'_R(B) \subset S'(B). \end{cases}$$

Here U is the unit disc centered at the origin and B is the leading coefficient, characterizing the function through the image of the origin: $B = F(O)$. The class notation repeats those of the normalized bounded

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univalent functions f:

$$\begin{cases} S(b) = \{f \mid f(z) = b(z + a_2 z^2 + \cdots),\, z \in U,\, |f(z)| < 1,\, 0 < b < 1\}, \\ S_R(b) \subset S(b). \end{cases}$$

Again, $S_R(b)$ means the real subclass of $S(b)$.

The observation on Śladkowska combined the above real classes together through the function L:

$$\begin{cases} L = L(z) = K^{-1}\left[\dfrac{4B}{(1-B)^2}\left(K(z) + \dfrac{1}{4}\right)\right], \\ K = K(z) = \dfrac{z}{(1-z)^2}. \end{cases}$$

Here K is the left Koebe-function and hence $L(U)$ is a unit disc with a left radial slit from the point -1 to the origin. The one-to-one correspondence

$$L \circ f \in S'_R(B), \quad L^{-1} \circ F \in S_R(b)$$

will be governed by aid of the development of L:

$$\begin{cases} y = L(z) = B + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \\ B_1 = \dfrac{4B(1-B)}{1+B}, \\ B_2 = \dfrac{8B(1-B)}{(1+B)^3}(1 - 2B - B^2), \\ B_3 = \dfrac{4B(1-B)}{(1+B)^5}(3 - 20B + 18B^2 + 12B^3 + 3B^4), \end{cases}$$

yielding

$$\begin{cases} b = \dfrac{A_1}{B_1}, \\ a_2 = \dfrac{A_2}{A_1} - \dfrac{B_2}{B_1^2}A_1, \\ a_3 = \dfrac{A_3}{A_1} - 2\dfrac{B_2}{B_1^2}A_2 + \left(2\dfrac{B_2^2}{B_1^4} - \dfrac{B_3}{B_1^3}\right)A_1^2. \end{cases}$$

The knowledge concerning the coefficient bodies of $S_R(b)$ allows determining the corresponding bodies of $S'_R(B)$ [2]. They are denoted by (A_2, A_1) and (A_3, A_2, A_1). For (A_2, A_1) we have

$$(A_2, A_1) = \left\{(A_1, A_2) \,\middle|\, -2A_1 + \frac{A_1^2}{B(1-B^2)} \le A_2 \le 2A_1 - \frac{2+B}{1-B^2}A_1^2, \right. \\ \left. 0 < A_1 < B_1 \right\}.$$

The body (A_3, A_2, A_1) is defined on (A_2, A_1) so that

$$E \le A_3 \le F,$$

where in the whole (A_2, A_1),

$$E = A_3 = \frac{A_2^2}{A_1} - A_1 + \frac{A_1^3}{(1-B^2)^2}.$$

The extremal domains connected to E are of left-right radial-slit types [2].

For F the area of definition is divided in three parts I, II and III visualized in Figure 1. The dividing lines I∩II and II∩III are determined by the limits

$$R^2\big[B_2 - 2B_1|\ln R|\big] \le A_2 \le R^2\big[B_2 + 2B_1|\ln R|\big],$$

where $R = A_1/B_1$.

The slit-type boundary functions extremizing F are similarly visualized in Figure 1.

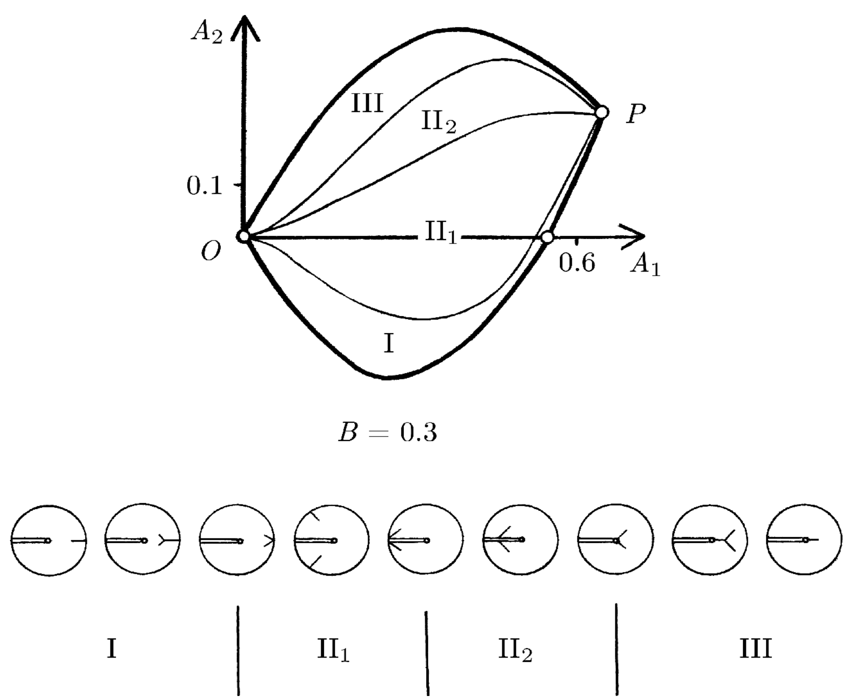

FIGURE 1

Observe that according to the extremal types the region II is split in two parts, II_1 and II_2 by the dividing line

$$A_2 = R^2\left[B_2 + 2B_1\frac{1-6B+B^2}{(1+B)^2}\ln R\right].$$

In the following denote

$$D_1 = B_3/B_1 - 2B_2^2/B_1^2.$$

By using this notation we have for F in the regions I and III (cf. [2]):

$$(1)\qquad \begin{cases} A_3 = \left[a_3 + 2\dfrac{B_2}{B_1^2}A_2 + D_1R^2\right]A_1 = F, \\ A_2 = A_1a_2 + B_2R^2, \\ a_2 = 2\delta(R - \sigma + \sigma\ln\sigma);\quad \sigma \in [R,1], \\ a_3 = 1 - R^2 + a_2^2 + 2\delta\cdot\sigma a_2 + 2(\sigma - R)^2. \end{cases}$$

Here $\delta = 1$ for I and $\delta = -1$ for III.

In II, F is defined by (cf. [2])

$$(2)\qquad \begin{cases} A_3 = \left[a_3 + 2\dfrac{B_2}{B_1^2}A_2 + D_1R^2\right]A_1 = F, \\ a_2 = \dfrac{A_2}{A_1} - \dfrac{B_2}{B_1}R, \\ a_3 = 1 - R^2 + \left(1 + \dfrac{1}{\ln R}\right)a_2^2. \end{cases}$$

2. Maximizing $A_1A_2A_3$. In some former papers, e.g. [3], a few simple functionals of the coefficients A_ν were considered. They were chosen to be independent of the sign of A_2. The present functional is free of that restriction. Thus

$$A_2 \geq 0:\ A_1A_2E \leq A_1A_2A_3 \leq A_1A_2F,$$
$$A_2 \leq 0:\ A_1A_2F \leq A_1A_2A_3 \leq A_1A_2E.$$

Consider first the local extremal point connected with A_1A_2E:

$$(3)\qquad A_2 \leq 0:\ Q = A_1A_2A_3 \leq A_1A_2E = A_2^3 + \left(\frac{A_1^4}{(1-B^2)^2} - A_1^2\right)A_2.$$

Differentiating this we obtain for the local extremal:

$$(4)\quad Q = \frac{\sqrt{3}}{36}(1-B^2)^3;\ A_1 = \frac{1-B^2}{\sqrt{2}},\ A_2 = -\frac{1-B^2}{2\sqrt{3}},\ A_3 = -\frac{\sqrt{2}}{6}(1-B^2).$$

The extremal point lies above the lower boundary arc ∂I of (A_2, A_1) if

$$-\frac{1-B^2}{2\sqrt{3}} - \left[-2A_1 + \frac{A_1^2}{B(1-B^2)}\right]_{A_1=\frac{1-B^2}{\sqrt{2}}} \geq 0$$
$$\Downarrow$$
$$(5)\qquad B \geq \widetilde{c} = \frac{6\sqrt{2}+\sqrt{3}}{23} = 0.444231834.$$

For the upper boundary arc ∂III of (A_2, A_1) we require

$$\left[2A_1 - \frac{2+B}{1-B^2}A_1^2\right]_{A_1=\frac{1-B^2}{\sqrt{2}}} \geq -\frac{1-B^2}{2\sqrt{3}},$$

which holds for the whole interval $0 < B < 1$.

For an interval below $\widetilde{c}$ the extremal point will be located on the lower boundary arc ∂I,

$$\partial\mathrm{I}: \ A_2 = -2A_1 + \frac{A_1^2}{B(1-B^2)},$$

where according to (3),

$$Q = -6A_1^3 + \frac{11A_1^4}{B(1-B^2)} - \frac{6+2B^2}{B^2(1-B^2)^2}A_1^5 + \frac{1+B^2}{B^3(1-B^2)^3}A_1^6.$$

For the local extremal point on ∂I we thus have

$$\begin{aligned} &-9[B(1-B^2)]^3 + 22[B(1-B^2)]^2A_1 \\ &\qquad - 5[B(1-B^2)](3+B^2)A_1^2 + 3(1+B^2)A_1^3 = 0. \end{aligned} \tag{6}$$

This condition is satisfied at the point (4) for $B = \widetilde{c}$.

Next, determine the local extremal point of $Q = A_1A_2F$ in the regions I and III. From (1) deduce

$$\begin{cases} \dfrac{1}{2A_1^3}\cdot\dfrac{\partial Q}{\partial\sigma} = h_0 + h_1A_1 + h_2A_1^2 = 0; \\ \qquad h_0 = \delta\ln\,\sigma(1+12s^2+12\sigma s+2\sigma^2), \\ \qquad h_1 = 4\ln\,\sigma(3s+\sigma)S, \\ \qquad h_2 = \delta\ln\,\sigma\big(13/B_1^2 + 12\delta B_2/B_1^3 + 2B_2^2/B_1^4 + B_3/B_1^3\big). \end{cases} \tag{7}$$

Further

$$\begin{cases} \dfrac{1}{A_1^2}\cdot\dfrac{\partial Q}{\partial A_1} = k_0 + k_1A_1 + k_2A_1^2 + k_3A_1^3 = 0; \\ \qquad k_0 = 6\delta s(1+4s^2+4\sigma s+2\sigma^2), \\ \qquad k_1 = 4(1+12s^2+4\sigma s+2\sigma^2)S, \\ \qquad k_2 = 10\delta s\big(2S^2 + 5/B_1^2 + 4\delta B_2/B_1^3 + B_3/B_1^3\big), \\ \qquad k_3 = 6\big(5/B_1^2 + 4\delta B_2/B_1^3 + B_3/B_1^3\big)S. \end{cases} \tag{8}$$

Here

$$s = \sigma\ln\,\sigma - \sigma, \ S = 2\delta/B_1 + B_2/B_1^2$$

and $\delta = 1$ for I and $\delta = -1$ for III.

From (7)

$$A_1 = \frac{-h_1 + \delta\cdot\sqrt{h_1^2 - 4h_0h_2}}{2h_2},$$

which, when substituted in (8), yields in the local extremal case σ and hence A_1, too.

There remains the maximizing of $Q = A_1 A_2 F$ in II. By aid of the abbreviations

$$A_1/B_1 = R,\ H = 1 + 1/\ln R;$$
$$D_2 = B_3/B_1 - B_2^2/B_1^2 - 1,\ D_3 = B_3/B_1 + 2B_2^2/B_1^2 - 1,$$

we obtain from (2)

$$\begin{cases} \dfrac{-\ln^2 R}{A_1 A_2} \cdot \dfrac{\partial Q}{\partial A_1} = a_2^2 + 4\dfrac{B_2}{B_1} R \ln\ R \cdot a_2 - 2\ln^2 R(1 + 2R^2 D_2), \\ \qquad \dfrac{1}{A_1^2} \cdot \dfrac{\partial Q}{\partial A_2} = 3Ha_2^2 + 2\dfrac{B_2}{B_1}(H+2)Ra_2 + 1 + D_3 R^2. \end{cases}$$

This yields the necessary extremal conditions for determining A_1 and A_2:

$$\begin{cases} 3Ha_2^2 + G_1 a_2 + G_2 = 0, \\ 3Ha_2^2 + G_3 a_2 + G_4 = 0, \end{cases}$$

$$\Updownarrow$$

$$(9)\qquad \begin{cases} a_2 = \dfrac{G_4 - G_2}{G_1 - G_3} \ \Rightarrow A_2 = A_1 a_2 + B_2 R^2, \\ 3Ha_2^2 + G_3 a_2 + G_4 = 0; \\ G_1 = 12H\dfrac{B_2}{B_1} R \ln R, \\ G_2 = -6H \ln^2 R(1 + 2R^2 D_2), \\ G_3 = 2\dfrac{B_2}{B_1}(H+2)R, \\ G_4 = 1 + D_3 R^2. \end{cases}$$

3. Maximalization results. In Table 1 there is a list of maximal points and values for increasing values of B. Observe, that the sign $-$ in the region-notation implies maximizing with negative A_2, i.e. the maximum is obtained from $A_1 A_2 E$ which means explicit expression (4) for $\max Q$. Similarly, $+$ indicates maximalization with positive A_2, from $A_1 A_2 F$, yielding results in implicit form.

There exist the following max max-cases:

$$\max\max Q = 0.037487883;\ B = b_1 = 0.105067336\ \in \mathrm{P},$$
$$\max\max Q = 0.026754453;\ B = b_2 = 0.397998215 \in \partial \mathrm{I}\,.$$

The maximizing point varies with increasing values of B. Crossing the boundaries between different regions of the body (A_3, A_2, A_1) occurs at the points c_2 and c_3:

$$B = c_2 = 0.185727645 \in \mathrm{II}_+ \cap \mathrm{III}_+,$$
$$B = c_3 = 0.453697122 \in \mathrm{I}_- \cap \mathrm{II}_-.$$

At

$$B = d = 0.312534879 \in \mathrm{III}_+, \partial\mathrm{I}$$

the maximalization occurs simultaneously on the upper surface III_+ and on the lower boundary $\partial\mathrm{I}$, determining at the same time

$$\min\max Q = 0.021714369;\ B = d \in \mathrm{III}_+\partial\mathrm{I}.$$

Such double maximal points may be called Twin Peaks on the surface of the coefficient body (A_3, A_2, A_1).

Table 1.

B	Region	A_1	A_2	A_3	$\max Q$
0.01	P	0.039208	0.075326	0.105567	0.000312
0.1	P	0.327273	0.427348	0.266517	0.037275
$0.105067 = b_1$	P	0.340353	0.434133	0.253711	0.037488
0.1051	P	0.340436	0.434173	0.253625	0.037488
$0.105369 = c_1$	$\mathrm{II}_+ \cap \mathrm{P}$	0.341122	0.434504	0.252918	0.037487
0.14	II_+	0.356935	0.412379	0.244326	0.035963
$0.185728 = c_2$	$\mathrm{II}_+ \cap \mathrm{III}_+$	0.355339	0.388176	0.233366	0.032189
0.2	III_+	0.350186	0.383550	0.230412	0.030947
0.3	III_+	0.312908	0.348136	0.208226	0.022683
$0.312535 = d$	III_+	0.308088	0.343354	0.205272	0.021714
$0.312535 = d$	$\partial\mathrm{I}$	0.455939	−0.174732	−0.272563	0.021714
0.35	$\partial\mathrm{I}$	0.495114	−0.192058	−0.262990	0.025008
0.38	$\partial\mathrm{I}$	0.522565	−0.205232	−0.247032	0.026493
0.39	$\partial\mathrm{I}$	0.530866	−0.209495	−0.240097	0.026702
$0.397998 = b_2$	$\partial\mathrm{I}$	0.537182	−0.212860	−0.233981	0.026754
0.4	$\partial\mathrm{I}$	0.538716	−0.213697	−0.232372	0.026751
$0.444232 = \widetilde{c}$	$\partial\mathrm{I} \cap \mathrm{I}_-$	0.567565	−0.231707	−0.189188	0.024880
0.45	I_-	0.563918	−0.230218	−0.187973	0.024403
$0.453697 = c_3$	$\mathrm{I}_- \cap \mathrm{II}_-$	0.561555	−0.229254	−0.187185	0.024098
0.46	II_-	0.557483	−0.227591	−0.185828	0.023578
0.5	II_-	0.530330	−0.216506	−0.176777	0.020297
0.6	II_-	0.452548	−0.184752	−0.150849	0.012612
0.7	II_-	0.360624	−0.147224	−0.120208	0.006382
0.8	II_-	0.254558	−0.103923	−0.084853	0.002245
0.9	II_-	0.134350	−0.054848	−0.044783	0.000330
0.99	II_-	0.014071	−0.005745	−0.004690	0.000000

The point $\widetilde{c}$ from (5) defines an interval $d \le B \le \widetilde{c}$ in which the maximizing point lies on $\partial\mathrm{I}$. From this onwards, in the interval $\widetilde{c} < B < 1$, the regions I_- or II_- take care of the maximalization.

If B is sufficiently close to 0 the point P assumes the role of the maximizing point. In order to find the shifting point $c_1 = \mathrm{II}_+ \cap \mathrm{P}$ let A_1 tend to B_1 so that

$$A_1 = B_1(1-h), \quad h \to +0.$$

From (9) we see that

$$a_2 = -\frac{B_1 \ln R}{2B_2}(1+D_3) + O(h),\ O(h) \to 0 \ \text{ for } \ h \to 0\,;$$

$$-\frac{1}{A_1 A_2} \cdot \frac{\partial Q}{\partial A_1} = K(B) + O(h),$$

where

$$K(B) = \frac{B_1^2}{4B_2^2}(1+D_3)^2 - 4D_2 - 2D_3 - 4.$$

Hence $\frac{\partial Q}{\partial A_1} = 0$ yields for $B = c_1$ the condition $K(B) = 0$, i.e.

(10) $$8B_1^2B_2^2 - 20B_1B_2^2B_3 + B_1^2B_3^2 + 4B_2^4 = 0$$

$$\Downarrow$$

$$B = c_1 = 0.105369060 \in \mathrm{II}_+ \cap \mathrm{P}.$$

The explicit part of the above estimation is collected as follows.

Result. *In $S'_R(B)$ the maximum of $A_1A_2A_3$ for the interval*

$$0.444031833 = \frac{6\sqrt{2}+\sqrt{3}}{23} = \widetilde{c} \le B < 1$$

occurs on the lower surface of the body (A_3, A_2, A_1):

$$\max A_1A_2A_3 = \frac{\sqrt{3}}{36}(1-B^2)^3,$$

at the point

$$A_1 = \frac{1-B^2}{\sqrt{2}},\ A_2 = -\frac{1-B^2}{2\sqrt{3}},\ A_3 = -\frac{\sqrt{2}}{6}(1-B^2).$$

In Figure 2 there is the graph connected with the values of the Table 1.

4. Minimalization results. According to the Section 2 the minimum of $Q = A_1A_1A_3$ is obtained from the expressions

$$A_1A_2E \text{ for } A_2 \geq 0,$$
$$A_1A_2F \text{ for } A_2 \leq 0.$$

Actually, only the last alternative will be realized. Therefore, the sign $-$, characterizing the region-notation, can be omitted.

Table 2.

B	Region	A_1	A_2	A_3	$\min Q$
0.05	∂I	0.034231	−0.044968	0.024882	−0.000038
0.1	∂I	0.262374	0.170606	−0.133010	−0.005954
0.2	∂I	0.489747	0.269737	−0.213725	−0.028234
0.27	∂I	0.612783	0.274543	−0.222069	−0.037360
$0.274376 = \beta_1$	∂I	0.619290	0.273003	−0.221185	−0.037395
0.28	∂I	0.627436	0.270719	−0.219810	−0.037337
$0.284717 = \gamma_1$	$\partial\text{I} \cap \text{P}$	0.634079	0.268541	−0.218451	−0.037197
0.285	P	0.634319	0.267964	−0.218773	−0.037186
$0.289393 = \gamma_2$	$\text{P} \cap \partial\text{III}$	0.637958	0.258988	−0.223541	−0.036934
0.29	∂III	0.637558	0.258804	−0.223569	−0.036890
0.3	∂III	0.630918	0.255757	−0.223967	−0.036140
0.4	∂III	0.559821	0.224215	−0.221370	−0.027786
$0.489950 = \delta$	∂III	0.489238	0.194240	−0.209355	−0.019895
$0.489958 = \delta$	I	0.308716	−0.325655	0.197891	−0.019895
0.5	I	0.314515	−0.327111	0.199710	−0.020547
$0.554728 = \gamma_3$	$\text{I} \cap \text{II}$	0.371011	−0.307904	0.207974	−0.023758
0.6	II	0.414995	−0.290090	0.218305	−0.026281
0.66	II	0.428346	−0.292403	0.223806	−0.028032
$0.667947 = \beta_2$	II	0.428169	−0.292795	0.223822	−0.028060
0.67	II	0.428053	−0.292886	0.223798	−0.028058
0.7	II	0.423061	−0.293516	0.222059	−0.027574
$0.790542 = \gamma_4$	$\text{II} \cap \text{P}$	0.369911	−0.278305	0.199329	−0.020521
0.8	P	0.355556	−0.272154	0.199590	−0.019313
0.9	P	0.189474	−0.169004	0.149698	−0.004794
0.99	P	0.019899	−0.019699	0.019500	−0.000008

There appears that the minimum may occur also on the upper boundary ∂III of (A_2, A_1);

$$\partial\mathrm{III}: \; A_2 = 2A_1 - \frac{2+B}{1-B^2}A_1^2$$

$$\Downarrow$$

$$\begin{aligned} Q &= A_1A_2E \\ &= 6A_1^3 - 11\frac{2+B}{1-B^2}A_1^4 + 2\frac{1+3(2+B)^2}{(1-B^2)^2}A_1^5 - \frac{2+B}{(1-B^2)^3}\big[1+(2+B)^2\big]A_1^6. \end{aligned}$$

Thus, for the local extremal point on ∂III there holds

$$\begin{aligned} &9(1-B^2)^2 - 22(2+B)(1-B^2)A_1 \\ &\qquad + 5\big[1+3(2+B)^2\big]A_1^2 - 3\frac{(2+B)\big[1+(2+B)^2\big]}{1-B^2}A_1^3 = 0. \end{aligned} \tag{11}$$

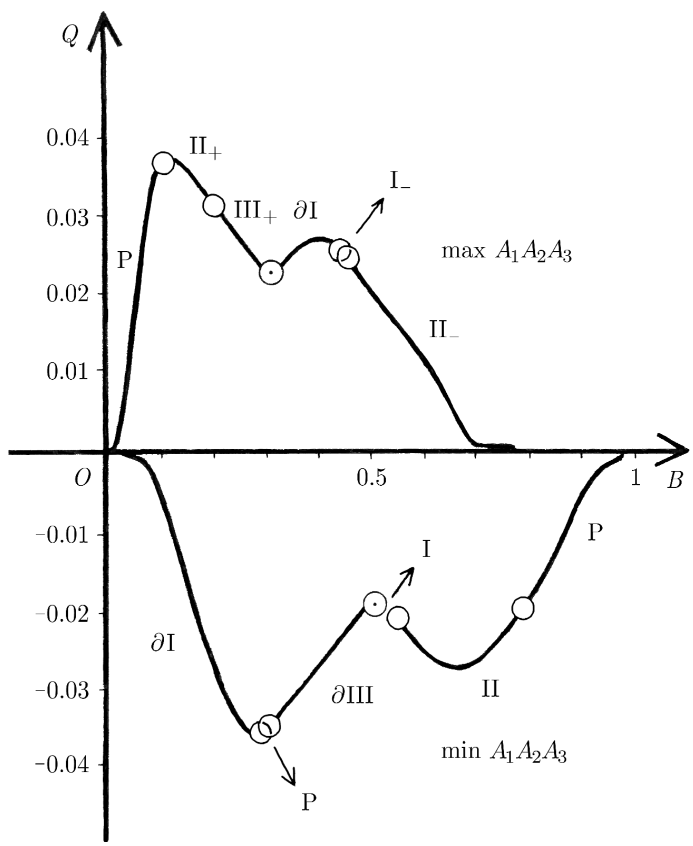

Figure 2

In Table 2 there is a collection of minimal points. Some of them deserve to be mentioned separately.

$$\min\min Q = -0.037395325;\ B = \beta_1 = 0.274376470 \in \partial\mathrm{I},$$
$$\min\min Q = -0.028059590;\ B = \beta_2 = 0.667947135 \in \mathrm{II}.$$

The tip P assumes the role of minimizing point three times. Shifting from ∂I to P occurs at $B = \gamma_1$. This point is found from (6) by aid of the limit process $A_1 \to B_1$, i.e. at (6) we have to take $A_1 = B_1$. Similarly, (11) with $A_1 = B_1$ yields the shifting point $B = \gamma_2$ from ∂III to P. At $B = \gamma_4$ we move from II to P by aid of (10). Between γ_2 and γ_4 there exists still another shifting point γ_3 of the type $\mathrm{I} \cap \mathrm{II}$. The results are:

$$\gamma_1 = 0.284716560 \in \partial\mathrm{I} \cap \mathrm{P},$$
$$\gamma_2 = 0.289392233 \in \mathrm{P} \cap \partial\mathrm{III},$$
$$\gamma_3 = 0.554728151 \in \mathrm{I} \cap \mathrm{II},$$
$$\gamma_4 = 0.790541920 \in \mathrm{II} \cap \mathrm{P}.$$

Finally, at

$$B = \delta = 0.489949658 \in \partial\mathrm{III},\ \mathrm{I}$$

there occur two simultaneous minima. We may speak about Twin Pits which, at the same time, happen to yield

$$\max\min Q = -0.019894996;\ B = \delta \in \partial\mathrm{III},\ \mathrm{I}.$$

The results of the Table 2 are visualized in Figure 2. In it the points of twin peaks and twin pits are pointed out by dotted circles.

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