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ANDRZEJ WIŚNICKI

An example of a nonexpansive mapping which is not 1-ball-contractive

Dedicated to W. A. Kirk on the occasion of his receiving an Honorary Doctorate from Maria Curie-Skłodowska University

Abstract. We give an example of an isometry defined on a convex weakly compact set which is not 1-ball-contractive. This gives an answer to an open question, implicitly included in Petryshyn (1975), and stated explicitly in Domínguez Benavides and Lorenzo Ramírez (2003, 2004). A fixed point theorem for multivalued contractions is also given.

Let X be a Banach space and C a nonempty bounded closed and convex subset of X. A mapping $T : C \to X$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in C,$$

and T is said to be 1-χ-contractive (or 1-ball-contractive) if

$$\chi(T(A)) \leq \chi(A)$$

for every $A \subset C$, where

$$\chi(A) = \inf \{d > 0 : A \text{ can be covered by finitely many balls of radii} < d\}$$

denotes the Hausdorff measure of noncompactness of a bounded set A.

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The problem of whether every nonexpansive mapping $T : C \to C$ is 1-χ-contractive implicitly appeared in [16, p. 235] and was explicitly addressed in [7, p. 384] and [8, p. 107], (in this direction, see also [12]), in the context of fixed point theory for multivalued nonexpansive mappings (see below). It was proved in [7, Th. 4.5] that such implication holds for weakly compact sets in separable or reflexive Banach spaces which satisfy the so-called nonstrict Opial condition.

The following example, partly inspired by [1, Remark 1.5.3], shows that this is not true in general, even for single-valued mappings.

Example. Let $X = C([0,1])$ be the Banach space of continuous functions defined on $[0,1]$ with the norm "supremum".

For $n = 1, 2, \ldots$, put $a_n(0) = a_n(1) = b_n(0) = b_n(1) = 0$,

$$a_n(t) = \begin{cases} 0 & \text{for } t = \frac{1}{10^n} \\ 1 & \text{for } t = \frac{2}{10^n} \\ 0 & \text{for } t = \frac{3}{10^n} \end{cases}, \qquad b_n(t) = \begin{cases} 0 & \text{for } t = \frac{4}{10^n} \\ 1 & \text{for } t = \frac{5}{10^n} \\ 0 & \text{for } t = \frac{6}{10^n} \\ -1 & \text{for } t = \frac{7}{10^n} \\ 0 & \text{for } t = \frac{8}{10^n} \end{cases}$$

and, by linear interpolation, define the functions a_n, b_n for all $t \in [0,1]$.

Let $A = \{\mathbf{0}, a_1, b_1, a_2, b_2, \ldots\}$ and $C = \operatorname{conv} A$. Set

$$T(\mathbf{0}) = \mathbf{0},\ T(a_n) = b_{2n-1},\ T(b_n) = b_{2n},\ n = 1, 2, \ldots$$

and

$$T(\alpha_1 x_1 + \cdots + \alpha_k x_k) = \alpha_1 T(x_1) + \cdots + \alpha_k T(x_k)$$

for $x_1, \ldots, x_k \in A,\ \alpha_1, \ldots, \alpha_k \geq 0,\ \alpha_1 + \cdots + \alpha_k = 1$.

It is not difficult to check that $T : C \to C$ is well defined,

$$\|Tx - Ty\| = \|x - y\|$$

for $x, y \in C$, and hence there exists a unique extension $\overline{T} : \overline{C} \to \overline{C}$ which is also an isometry.

Moreover $\overline{C}$ is weakly compact because the set A is. Indeed, for every sequence $\{x_n\} \subset A$ there exists a subsequence $\{x_{n_k}\}$ which is constant or tends pointwise to $\mathbf{0}$ and consequently $\{x_{n_k}\}$ is weakly convergent to a point in A.

On the other hand (see for instance [1], [3]),

$$\chi(\{a_1, a_2, \ldots\}) = \frac{1}{2} \lim_{h \to 0^+} \sup_{x \in \{a_1, a_2, \ldots\}} \sup_{\|t-s\| \leq h} |x(t) - x(s)| = \frac{1}{2}$$

and similarly

$$\chi\left(\overline{T}(\{a_1, a_2, \ldots\})\right) = 1.$$

Hence $\overline{T} : \overline{C} \to \overline{C}$ is only 2-χ-contractive and the problem is solved. Notice that $\overline{T}$ is an affine mapping.

Remark. It is not very clear if there exists a similar mapping defined on the whole unit ball.

The notions of k-Lipschitzian and k-χ-contractive mappings are easily generalized to the multivalued case.

Let $CB(X)$ denote the family of all nonempty bounded closed subsets of X, $K(X)$ the family of nonempty compact subsets of X and $KC(X)$ the family of all nonempty compact convex subsets of X. For $A, B \in CB(X)$, the Hausdorff metric is given by

$$H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}.$$

A multivalued mapping $T : C \to CB(X)$ is said to be k-Lipschitzian if

$$H(Tx,Ty) \le k\|x-y\|, \quad x,y \in C,$$

and T is is said to be χ-condensing (respectively, k-χ-contractive) if, for each bounded subset A of C with $\chi(A) > 0$, $T(A)$ is bounded and

$$\chi(T(A)) < \chi(A) \qquad (\text{respectively, } \chi(T(A)) \le k\chi(A)).$$

Here $T(A) = \bigcup_{x\in A} Tx$. A k-Lipschitzian mapping is called a contraction if $k < 1$.

Recall that a multivalued mapping $T : C \to 2^X$ is upper semicontinuous if $\{x \in C : Tx \subset V\}$ is open in C whenever $V \subset X$ is open. The inward set of C at $x \in C$ is defined by

$$I_C(x) = \{x + \lambda(y-x) : \lambda \ge 0, y \in C\}.$$

Theorem 1 (Deimling [5], see also Reich [17])**.** *Let C be a bounded closed convex subset of a Banach space X and let $T : C \to KC(X)$ be upper semicontinuous and χ-condensing. If $Tx \cap \overline{I_C(x)} \ne \emptyset$ for all $x \in C$, then T has a fixed point.*

In [19] (see also [18]), Hong-Kun Xu applied Theorem 1 to extend the Kirk theorem [13], see also [14], [15]. (In fact he used a less general result due to Browder [4], see also [11].) Soon after, new results were obtained by Domínguez Benavides and Lorenzo Ramírez [7], [8], [9].

The above example shows that the limitations of the use of Theorem 1 in fixed point theory for multivalued nonexpansive mappings are rather fundamental. It was shown in [9], how to overcome these limitations in the important case of nonexpansive "self-mappings", see [9, Th. 3.3]. The key observation was to use χ_C rather than χ. Recall that

$$\chi_C(A) = \inf\{d > 0 : A \text{ can be covered by finitely many balls with centers in } C \text{ and of radii} < d\}$$

is called the relative Hausdorff measure of noncompactness of A with respect to C. Below we give a mild generalization of that result.

The following lemma simplifies and generalizes Theorem 3.2 in [9].

Lemma 2. *Let C be a closed subset of a Banach space X. Assume that $T : C \to K(C)$ is a k-Lipschitzian multivalued mapping. Then T is k-χ_C-contractive.*

Proof. Let A be a nonempty bounded subset of C, $\varepsilon > 0$ and fix $x_1, x_2, \ldots, x_k \in C$ such that

$$A \subset \bigcup_{i=1}^{k} B(x_i, \chi_C(A) + \varepsilon). \tag{1}$$

Since T has compact values, we can choose $y_1, y_2, \ldots, y_n \in C$ such that

$$\bigcup_{i=1}^{k} Tx_i \subset \bigcup_{j=1}^{n} B(y_j, \varepsilon). \tag{2}$$

It follows from (1) and from the definition of the Hausdorff metric that for every $x \in A$ and $y \in Tx$ there exist $i \in \{1, 2, \ldots, k\}$ and $z \in Tx_i$ such that

$$\|y - z\| \le H(Tx, Tx_i) + \varepsilon \le k\|x - x_i\| + \varepsilon \le k(\chi_C(A) + \varepsilon) + \varepsilon.$$

Moreover, by (2), there exists $j \in \{1, 2, \ldots, n\}$ such that $\|z - y_j\| \le \varepsilon$. Hence

$$\|y - y_j\| \le k(\chi_C(A) + \varepsilon) + 2\varepsilon$$

and consequently $\chi_C(T(A)) \le k\chi_C(A)$. □

We are now in a position to prove the following theorem which generalizes Theorem 3.3 in [9].

Theorem 3. *Let C be a closed convex subset of a Banach space X, $T : C \to KC(C)$ be a multivalued contraction and let D be a bounded closed convex subset of C. If $Tx \cap \overline{I_D(x)} \neq \emptyset$ for all $x \in D$, then T has a fixed point in D.*

Proof. It suffices to notice that χ_C shares basic properties of (classic) measures of noncompactness, use Lemma 2 and follow the proof of [5, Th. 1], (see also [6, p. 153]). □

We conclude with the following conjecture which naturally arises in view of Theorems 1 and 3.

Conjecture. Let C be a bounded closed convex subset of a Banach space X and let $T : C \to KC(X)$ be a multivalued contraction. If $Tx \cap \overline{I_C(x)} \neq \emptyset$ for all $x \in C$, then T has a fixed point.

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References

[1] Akhmerov, R. R., M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina and B. N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*, Nauka, Novosybirsk, 1986, (Russian). English edition: Birkhäuser Verlag, Basel, 1992.

[2] Ayerbe, J. M., T. Domínguez Benavides and G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser Verlag, Basel, 1997.

[3] Banaś, J., K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[4] Browder, F. E., *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.

[5] Deimling, K., *Fixed points of weakly inward multis*, Nonlinear Anal. **10** (11) (1986), 1261–1262.

[6] Deimling, K., *Multivalued Differential Equations*, Walter de Gruyter, Berlin, 1992.

[7] Domínguez Benavides, T., P. Lorenzo-Ramírez, *Fixed point theorems for multivalued nonexpansive mappings without uniform convexity*, Abstr. Appl. Anal. **2003:6** (2003), 375–386.

[8] Domínguez Benavides, T., P. Lorenzo-Ramírez, *Fixed point theorems for multivalued nonexpansive mappings satisfying inwardness conditions*, J. Math. Anal. Appl. **291** (2004), 100–108.

[9] Domínguez Benavides, T., P. Lorenzo-Ramírez, *Asymptotic centers and fixed points for multivalued nonexpansive mappings*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **58** (2004), 37–45.

[10] Goebel, K., W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.

[11] Halpern, B., *Fixed point theorems for set-valued maps in infinite-dimensional spaces*, Math. Ann. **189** (1970), 87–98.

[12] Kamenskii, M., V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, Walter de Gruyter, Berlin, 2001.

[13] Kirk, W. A., *Nonexpansive mappings in product spaces, set-valued mappings and k-uniform rotundity*, Nonlinear Functional Analysis and Applications (F. E. Browder, ed.), (Proc. Sympos. Pure Math., Vol. 45, Part 2) American Mathematical Society, Rhode Island, 1986, pp. 51–64.

[14] Kirk, W. A., S. Massa, *Remarks on asymptotic and Chebyshev centers*, Houston J. Math. **16** (1990), 357–364.

[15] Kuczumow, T., S. Prus, *Compact asymptotic centers and fixed fixed points of multivalued nonexpansive mappings*, Houston J. Math. **16** (1990), 465–468.

[16] Petryshyn, W. V., *On the approximation-solvability of equation involving A-proper and pseudo-A-proper mappings*, Bull. Amer. Math. Soc. **81** (1975), 222–312.

[17] Reich, S., *Fixed points in locally convex spaces*, Math. Z. **125** (1972), 17–31.

[18] Xu, H. K., *Metric fixed point theory for multivalued mappings*, Dissertationes Math. (Rozprawy Mat.) **389** (2000), 1–39.

[19] Xu, H. K., *Multivalued nonexpansive mappings in Banach spaces*, Nonlinear Anal. **43** (2001), no. 6, 693–706.

Andrzej Wiśnicki
Institute of Mathematics
Maria Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 1
20-031 Lublin, Poland
e-mail: awisnic@golem.umcs.lublin.pl

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