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Tensor fields on LM induced by tensor fields on M by means of connections on M

Abstract. We describe all natural operators $\mathcal{A}$ transforming a classical linear connection ∇ on an m-dimensional manifolds M and a tensor field t of type (r,s) on M into a tensor field $\mathcal{A}(\nabla,t)$ of type (p,q) on the frame bundle LM over M.

0. Introduction. There are some equivalent notions of classical linear connections, [1]–[3]. They are covariant derivatives ∇ satisfying well-known properties, they are systems of Christoffel symbols Γ^i_{jk} with the well-known transformation rules, they are homothety invariant distributions H on the tangent bundle such that $TTM = VTM \oplus H$, they are so-called horizontal liftings $(\ )^H$ to the tangent bundle, they are fiber linear sections $\lambda : TM \to J^1TM$ of the first jet prolongation of the tangent bundle, and they are sections of so-called connection bundle QM. (We recall that the connection bundle on a manifold M is $QM = \pi^{-1}(id_{TM}) \subset T^*M \otimes J^1TM$, where $\pi : T^*M \otimes J^1TM \to T^*M \otimes TM$ is the usual projection).

In [2, Theorem 33.16], it is described how a classical linear connection ∇ on an m-dimensional manifold M and a tensor field t of type (r,s) on M induce tensor field $\mathcal{A}(\nabla,t)$ of type (p,q) on M, provided $r < s$. More precisely, there are classified all respective natural operators.

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In the present paper, we study how a classical linear connection ∇ on an m-dimensional manifold M and a tensor field t of type (r,s) on M can induce a tensor field $\mathcal{A}(\nabla,t)$ on the linear frame bundle LM over M. This problem is reflected in the concept of $\mathcal{M}f_m$-natural operators $\mathcal{A}: Q\times T^{(r,s)} \rightsquigarrow T^{(p,q)}L$, where $\mathcal{M}f_m$ is the category of all m-dimensional manifolds and their embeddings. We describe all natural operators $\mathcal{A}$ in question.

We recall that an $\mathcal{M}f_m$-natural operator $\mathcal{A}: Q\times T^{(r,s)} \rightsquigarrow T^{(p,q)}L$ in the sense of [2] is a regular and $\mathcal{M}f_m$-invariant system of operators

$$\mathcal{A}:\Gamma(QM)\times\Gamma(T^{(r,s)}M)\to\Gamma(T^{(p,q)}(LM))$$

for any manifold M, where $\Gamma(QM)$ is the set of sections of $QM\to M$ (classical linear connections on M), $\Gamma(T^{(r,s)}M)$ is the space of all tensor fields of type (r,s) on M and $\Gamma(T^{(p,q)}(LM))$ is the space of all tensor fields of type (p,q) on LM. The invariance of $\mathcal{A}$ means that for any $\mathcal{M}f_m$-map $\varphi: M\to N$ if connections ∇_1 on M and ∇_2 on N are φ-related and tensor fields τ_1 and τ_2 of type (r,s) on M and N are φ-related then tensor fields $\mathcal{A}(\nabla_1,\tau_1)$ and $\mathcal{A}(\nabla_2,\tau_2)$ of type (p,q) on LM and LN are $L\varphi$-related, where $L\varphi: LM\to LN$ is the induced fibered map. The regularity of $\mathcal{A}$ means that $\mathcal{A}$ transforms smoothly parametrized families of sections into smoothly parametrized families of sections.

From now on $x^1,\dots,x^m$ is the usual coordinate system on $\mathbf{R}^m$. All manifolds and maps are assumed to be smooth (of class $\mathcal{C}^\infty$).

1. The $\mathcal{M}f_m$-natural operators $Q\times T^{(r,s)} \rightsquigarrow T^{(0,0)}L$ of finite order. Let $\theta=(\frac{\partial}{\partial x^1}|_0,\dots,\frac{\partial}{\partial x^m}|_0)\in L_0\mathbf{R}^m$ be the frame. Let S^k be the vector space of all k-jets at $0\in\mathbf{R}^m$ of classical linear connections ∇ on $\mathbf{R}^m$ given by the Christoffel symbols $\Gamma^i_{jl}:\mathbf{R}^m\to\mathbf{R}$ satisfying

$$\sum_{j,l=1}^m \Gamma^i_{jl}(x)x^jx^l=0 \text{ for } i=1,\dots,m\ .$$

Equivalently, S^k is the space of all k-jets at 0 of classical linear connections ∇ on $\mathbf{R}^m$ such that the usual coordinate system $x^1,\dots,x^m$ on $\mathbf{R}^m$ is a normal coordinate system for ∇ with center 0.

Let us consider a smooth function

$$\mu: S^k\times J^k_0T^{(r,s)}\mathbf{R}^m\to\mathbf{R}.$$

Given a classical linear connection ∇ on an m-manifold M and a tensor field t of type (r,s) on M we define a smooth map $\mathcal{B}^{<\mu>}(\nabla,t): LM\to\mathbf{R}$ by

$$\mathcal{B}^{<\mu>}(\nabla,t)(\sigma):=\mu\big(j^k_0(\varphi_*\nabla),j^k_0(\varphi_*t)\big)$$

for $\sigma\in(LM)_x$, $x\in M$, where φ is a normal coordinate system on M for ∇ with center x such that $\varphi(x)=0$ and $L\varphi(\sigma)=\theta$. The definition is correct because $germ_x(\varphi)$ is determined uniquely. (Indeed, for another such normal coordinate system φ_1 we have $\varphi_1 = A \circ \varphi$ near x for some $A \in Gl(m)$ (see [1]) with $LA(\theta) = \theta$. So, $A = id$ and $\varphi_1 = \varphi$ near x, as well.)

The correspondence $\mathcal{B}^{<\mu>} : Q \times T^{(r,s)} \rightsquigarrow T^{(0,0)}L$ is an $\mathcal{M}f_m$-natural operator of order k. It can be proved as follows.

Let $j^k_x\nabla_1 = j^k_x\nabla_2$, $j^k_x t_1 = j^k_x t_2$, where $x \in M$, $\sigma \in L_xM$. Then $j^{k+2}_{0_x} Exp^{\nabla_1} = j^{k+2}_{0_x} Exp^{\nabla_2}$ (it is well-known fact). Then $j^{k+2}_x\varphi_1 = j^{k+2}_x\varphi_2$, where φ_1, φ_2 are the unique normal coordinates for ∇_1 and ∇_2 determined by σ in question.

Then $j^k_0((\varphi_1)_*\nabla_1) = j^k_0((\varphi_2)_*\nabla_2)$ and $j^k_0((\varphi_1)_*t_1) = j^k_0((\varphi_2)_*t_2)$. Then $\mathcal{B}^{<\mu>}(\nabla_1, t_1)(\sigma) = \mathcal{B}^{<\mu>}(\nabla_2, t_2)(\sigma)$ as well.

Proposition 1. *Any $\mathcal{M}f_m$-natural operator $\mathcal{B} : Q \times T^{(r,s)} \rightsquigarrow T^{(0,0)}L$ of finite order k is equal to $\mathcal{B}^{<\mu>}$ for some unique smooth function $\mu : S^k \times J^k_0 T^{(r,s)}\mathbf{R}^m \to \mathbf{R}$.*

Proof. Let $\mathcal{B}$ be an operator in question. Define $\mu : S^k \times J^k_0 T^{(r,s)}\mathbf{R}^m \to \mathbf{R}$ by

$$\mu(j^k_0(\nabla), j^k_0 t) = \mathcal{B}(\nabla, t)_\theta.$$

Clearly, $\mathcal{B} = \mathcal{B}^{<\mu>}$. □

2. Some vector fields on LM from a connection on M. Let ∇ be a classical linear connection on M. For any $\xi \in \mathbf{R}^m$ we have the fundamental horizontal vector field $B^\xi(\nabla)$ on LM defined by $T\pi(B^\xi(\nabla)_l) = l(\xi)$, $l \in LM$ and $\pi : LM \to M$ is the bundle projection.

For any $A \in gl(m)$ we have the fundamental vertical vector field A^* on LM. We have the following well-known fact.

Proposition 2. *Let e_i be the usual basis in $\mathbf{R}^m$ and E^j_l be the usual basis in $gl(m)$. Given a classical linear connection ∇ on M the vector fields $B^{e_i}(\nabla)$ and $(E^j_l)^*$ for $i, j, l = 1, \dots, m$ form the basis over $\mathcal{C}^\infty(LM)$ of vector fields on LM.*

3. The $\mathcal{M}f_m$-natural operators $\mathcal{A} : Q \times T^{(r,s)} \rightsquigarrow T^{(p,q)}L$ of finite order. The space of all $\mathcal{M}f_m$-natural operators $\mathcal{A} : Q \times T^{(r,s)} \rightsquigarrow T^{(p,q)}L$ of finite order k is (in obvious way) the module over the algebra of all (classified in Section 1) $\mathcal{M}f_m$-natural operators $B : Q \times T^{(r,s)} \rightsquigarrow T^{(0,0)}L$ of order k.

Proposition 3. *The module of all $\mathcal{M}f_m$-natural operators $\mathcal{A} : Q \times T^{(r,s)} \rightsquigarrow T^{(p,q)}L$ of order $k < \infty$ is free and finite dimensional. Let $F^a(\nabla)$ be the basis of tensor fields on LM of type (p, q) obtained from the basis $(B^{e_i}(\nabla), (E^j_l)^*)$ by the dualization and tensoring. Then the (constant in the second factor) $\mathcal{M}f_m$-natural operators $F^a : Q \times T^{(r,s)} \rightsquigarrow T^{(p,q)}L$ form the basis in the module in question.*

Proof. Let $\mathcal{C} : Q \times T^{(r,s)} \rightsquigarrow T^{(p,q)}L$ be an $\mathcal{M}f_m$-natural operator of order k. For any classical linear connection ∇ on M and any tensor field t of type

(r,s) on M we can write

$$\mathcal{A}(\nabla, t) = \sum_a \lambda_a(\nabla, t) F^a(\nabla),$$

where $\lambda_a(\nabla, t) : LM \to \mathbf{R}$ are the uniquely determined maps.

Because of the invariance od $\mathcal{A}$ with respect to $\mathcal{M}f_m$-maps, $\lambda_a : Q \times T^{(r,s)} \rightsquigarrow T^{(0,0)}L$ are $\mathcal{M}f_m$-natural operators. $\square$

4. The infinite order case. For $k = \infty$, the results are similar. We need only replace $\mu : S^k \times J^k_0 T\mathbf{R}^m \to \mathbf{R}$ by smooth $\mu : S^\infty \times J^\infty_0 T\mathbf{R}^m \to \mathbf{R}$. The smoothness means that μ is locally factorized by smooth maps $S^k \times J^k_0 T\mathbf{R}^m \to \mathbf{R}$ with finite k. In the proof of (new) Proposition 1, the additional assumption on μ is obtained by the non-linear Petree theorem, [2].

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