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Harmonic univalent functions convex in orthogonal directions

Abstract. Many extremal problems in the classes S_H and S_H^0 of normalized univalent harmonic mappings in the unit disk such as coefficient estimates are still opened. However, most of these estimates are conjectured and have been proved for over twenty years in some subclasses of typically real functions, starlike functions, close-to-convex functions, or functions convex in one direction, etc. On the other hand, there is, probably most known and best examined, the subclass of convex functions, in which estimates are completely different from those written above. We introduce new subclasses, by the geometric condition of convexity in two orthogonal directions, in particular, directions of the axis and establish some estimates for them. Obtained results are settled between those proved for convex functions and conjectured in the full classes.

1. Introduction. Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$. A complex-valued harmonic function $f : \Delta \to \mathbb{C}$ has the representation

$$f = h + \overline{g}, \tag{1.1}$$

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where h and g are analytic in Δ. Hence, they have the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \Delta,$$

where $a_n, b_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$. Choose $g(0) = 0$ (i.e. $b_0 = 0$) so the representation (1.1) is unique in Δ and is called the canonical representation of f.

For univalent and sense-preserving harmonic functions f in Δ, it is convenient to make further normalization (with no loss of generality) $h(0) = 0$ (i.e. $a_0 = 0$) and $h'(0) = 1$ (i.e. $a_1 = 1$). The family of all such functions f is denoted by S_H. The family of all functions $f \in S_H$ with the additional property that $g'(0) = 0$ (i.e. $b_1 = 0$) is denoted by S_H^0. Observe, that the classical family S consists of all functions $f \in S_H^0$ such that $g(z) \equiv 0$. Thus, it is clear that $S \subset S_H^0 \subset S_H$.

Now recall, that a domain $D \subset \mathbb{C}$ is said to be convex in one direction of the line $z = te^{i\theta}$, $t \in \mathbb{R}$, for a given constant $\theta \in [0, \pi)$, if $D \cap \{z_0 + te^{i\theta} : t \in \mathbb{R}\}$ is a connected set (or empty), for each $z_0 \in \mathbb{C}$.

We introduce classes $COD_H(\theta)$ and $COD_H^0(\theta)$ of all functions $f \in S_H$ and $f \in S_H^0$, respectively, such that $f(\Delta)$ is convex in two directions of the lines $z = te^{i\theta}$, $t \in \mathbb{R}$ and $z = te^{i(\theta+\pi/2)}$, $t \in \mathbb{R}$, for each $\theta \in [0, \pi/2)$. Now we are ready to define classes

$$CAD_H := COD_H(0), \quad COD_H := \bigcup_{\theta \in [0, \frac{\pi}{2})} COD_H(\theta),$$

$$CAD_H^0 := COD_H^0(0), \quad COD_H^0 := \bigcup_{\theta \in [0, \frac{\pi}{2})} COD_H^0(\theta).$$

Note that we have simple relation between CAD_H and COD_H. Likewise, we have the same relation between CAD_H^0 and COD_H^0.

Remark 1.1. For every function $F \in COD_H$ there exists function $f \in CAD_H$ so that $F(z) = e^{i\theta} f\big(e^{-i\theta} z\big)$, where $\theta \in [0, \frac{\pi}{2})$ is some constant.

In this paper we provide solutions to some extremal problems in COD_H and COD_H^0 such as coefficient, distortion, and growth estimates.

2. Backgrounds and examples. To prove main results of this paper several known theorems are needed. We recall them now without proofs.

Theorem 2.1 (Bieberbach–de Branges Theorem, [2]). *If $f \in S$ then*

$$|a_n| \le n, \quad n = 2, 3, 4, \ldots.$$

Furthermore, if the equality holds for some n then the function f is the Koebe function $k_0(z) := z(1-z)^{-2}$, $z \in \Delta$ or its rotation.

Theorem 2.2 (Distortion Theorem, [5, Chapter 2, Theorem 2.5])**.** *If $f \in S$ then*

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3},$$

where $r := |z|$ and $z \in \Delta$. Equality holds only for the Koebe function and its rotations.

Theorem 2.3 (Growth Theorem, [5, Chapter 2, Theorem 2.6])**.** *If $f \in S$ then*

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2},$$

where $r := |z|$ and $z \in \Delta$. Equality holds only for the Koebe function and its rotations.

The basic tool that we use in this paper is a slight generalization of the Clunie and Sheil-Small's shear construction theorem (see [1], [6]). It shows how to apply a part of classical theory of conformal mappings to harmonic functions convex in the directions of the axis.

Theorem 2.4. *Let $\theta \in [0, \pi)$ and let f be a harmonic and locally univalent function in Δ satisfying* (1.1)*. Then f is univalent and $f(\Delta)$ is a set convex in the direction of the line $z = te^{i\theta}$, $t \in \mathbb{R}$, if and only if the analytic function $h - e^{2i\theta} g$ is univalent and $\big(h - e^{2i\theta} g\big)(\Delta)$ is a set convex in the same direction.*

Theorem 2.4, in the cases where $\theta = 0$ and $\theta = \pi/2$, can be used as a starting point in constructing harmonic functions f in Δ such that $f(\Delta)$ is a set convex in the directions of the axis, in particular, functions from CAD_H and CAD_H^0.

We give an example of a conformal mapping $S \ni f : \Delta \to \mathbb{C}$, such that $f(\Delta) = \Omega$, where Ω is an angle in the complex plane $\mathbb{C}$ of given measure $3\pi/2$. Obviously, this function is convex in orthogonal directions. Moreover, it seems to be extremal in many problems concerning such conformal mappings.

Example 2.5. Let $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}\{z\} > 0\}$. Consider the mappings

$$f_1 : \Delta \to \mathbb{C}^+, \quad f_1(z) := i\frac{\overline{a}z + a}{1 - z}, \quad \mathrm{Re}\{a\} > 0$$

and

$$f_2 : \mathbb{C}^+ \to \Omega, \quad f_2(z) := z^{3/2}.$$

The composition of f_1 and f_2 with suitable normalization gives

$$S \ni f : \Delta \to \Omega, \quad f(z) := \frac{(\frac{\overline{a}z+a}{1-z})^{3/2} - a^{3/2}}{3\,\mathrm{Re}\{a\} a^{1/2}}, \quad \mathrm{Re}\{a\} > 0.$$

In particular, the modulus of the second coefficient in the power series expansion of f is maximal and equals $3/2$, when we choose $\operatorname{Im}\{a\} = 0$. Then the function f is of the form

$$f(z) = \frac{1}{3}\left[\left(\frac{z+1}{1-z}\right)^{3/2} - 1\right].$$

It is interesting to give an example of a harmonic function f, which is convex in the directions of the axis but neither f is convex nor even starlike.

Example 2.6. Consider the conformal mapping φ of Δ onto equilateral triangle given by the Schwarz–Christoffel formula as follows (see [8])

$$\varphi(z) := \int_0^z (1-\zeta^3)^{-2/3}\,\mathrm{d}\,\zeta.$$

Let f be of the form (1.1), such that $h + g = \varphi$ and $g'/h' = z^3$. We may determine f (see Figure 1) computing

$$h(z) = \int_0^z \frac{(1-\zeta^3)^{-2/3}}{1+\zeta^3}\,\mathrm{d}\,\zeta \tag{2.1}$$

and

$$g(z) = \int_0^z \frac{\zeta^3(1-\zeta^3)^{-2/3}}{1+\zeta^3}\,\mathrm{d}\,\zeta. \tag{2.2}$$

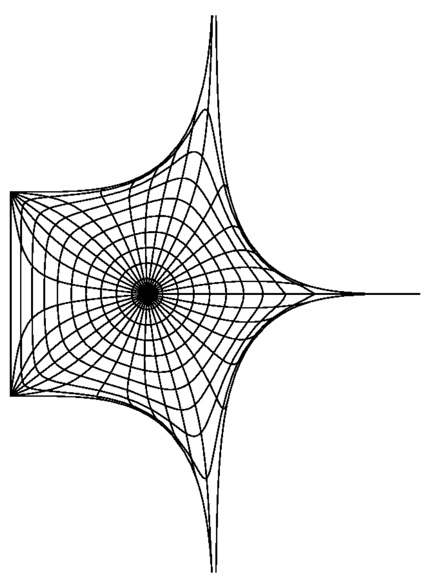

FIGURE 1. The image of the mapping given by (2.1)–(2.2).

Theorem 2.4 implies that the function f is univalent and convex in the vertical direction. Now we prove that f is convex in the horizontal direction by proving that the function $\tilde{f} : z \mapsto -i[h(iz) - g(iz)]$ is convex in the vertical direction. We use the necessary and sufficient condition due to Royster and Ziegler (see [7])

$$\operatorname{Re}\big\{-ie^{i\mu}\big(1 - 2e^{-i\mu}\cos\nu z + e^{-2i\mu}z^2\big)\tilde{f}'(z)\big\} \geq 0, \quad z \in \Delta,$$

where $\mu, \nu \in [0, \pi]$ are some constants. When we choose $\mu = \pi/2$ and $\nu = 2\pi/3$ then the condition is equivalent to

$$\operatorname{Re}\left\{\frac{(1+z^3)^{1/3}}{1-z}\right\} \geq 0, \quad z \in \Delta. \tag{2.3}$$

First, observe that if $z = e^{it}$, $t \in [-\pi, 0) \cup (0, \pi]$ then we have

$$\begin{aligned}\operatorname{Re}\left\{\frac{(1+z^3)^{1/3}}{1-z}\right\} &= \frac{\operatorname{Re}\{(1+z^3)^{1/3}(1-\bar{z})\}}{|1-z|^2}\\ &= \frac{2^{1/3}3^{1/2}}{|1-z|^2}\begin{cases}0, & t \in [-\pi/3, \pi/3],\\ -(\cos 3t/2)^{1/3}\sin t/2 \geq 0, & t \in [-\pi, -\pi/3],\\ (\cos 3t/2)^{1/3}\sin t/2 \geq 0, & t \in [\pi/3, \pi].\end{cases}\end{aligned}$$

Next, observe that if $z = re^{it}$, $t \in [0, \pi/3]$, $r \in [0, 1)$ then we have

$$0 \leq \operatorname{Arg}\{(1+z^3)^{1/3}\} \leq \operatorname{Arg}\{(1+e^{3it})^{1/3}\} = t/2 \leq \pi/6$$

and

$$0 \leq \operatorname{Arg}\{1-\bar{z}\} \leq \operatorname{Arg}\{1-e^{-it}\} = \pi/2 - t/2 \leq \pi/2.$$

Hence, we may write

$$\begin{aligned}0 &\leq \operatorname{Arg}\{(1+z^3)^{1/3}(1-\bar{z})\} = \operatorname{Arg}\{(1+z^3)^{1/3}\} + \operatorname{Arg}\{1-\bar{z}\}\\ &\leq \operatorname{Arg}\{(1+e^{3it})^{1/3}\} + \operatorname{Arg}\{1-e^{-it}\}\\ &= \operatorname{Arg}\{(1+e^{3it})^{1/3}(1-e^{-it})\} \leq \pi/2.\end{aligned}$$

Since $\operatorname{Re}\{\alpha\} = \operatorname{Re}\{\bar{\alpha}\}$ for any $\alpha \in \mathbb{C}$, then we obtain that (2.3) holds for all $z = re^{it}$, $t \in [-\pi/3, \pi/3]$, $r \in [0, 1)$. Finally, the minimum principle for harmonic functions implies that (2.3) holds also for all $z = re^{it}$, $t \in [-\pi, -\pi/3) \cup (\pi/3, \pi]$, $r \in [0, 1)$.

To show that f is not starlike observe, that

$$\operatorname{Re}\{f(i)\} = \operatorname{Re}\left\{\int_0^i (1-\zeta^3)^{-2/3}\,\mathrm{d}\,\zeta\right\} = -\operatorname{Im}\left\{\int_0^1 \frac{(1-ir^3)^{2/3}}{|1+ir^3|^{4/3}}\,\mathrm{d}\,r\right\} > 0,$$

$$\operatorname{Im}\{f(i)\} = \operatorname{Im}\left\{\int_0^i \frac{(1-\zeta^3)^{1/3}}{1+\zeta^3}\,\mathrm{d}\,\zeta\right\} = \operatorname{Re}\left\{\int_0^1 \frac{(1+ir^3)^{4/3}}{|1-ir^3|^2}\,\mathrm{d}\,r\right\} > 0$$

and

$$\begin{aligned}\operatorname{Re}\{f(e^{\pi/3})\} &= \operatorname{Re}\{f(i)\} - \operatorname{Re}\left\{\int_{e^{\pi/3}}^{i} (1-\zeta^3)^{-2/3}\,\mathrm{d}\,\zeta\right\}\\ &= \operatorname{Re}\{f(i)\} + \operatorname{Im}\left\{\int_{\pi/3}^{\pi/2} \frac{e^{it}(1-e^{-3it})^{2/3}}{|1-e^{3it}|^{4/3}}\,\mathrm{d}\,t\right\}\\ &> \operatorname{Re}\{f(i)\} > 0,\\ \operatorname{Im}\{f(e^{\pi/3})\} &= \operatorname{Im}\{f(i)\} - \operatorname{Im}\left\{\int_{0}^{i} \frac{(1-\zeta^3)^{1/3}}{1+\zeta^3}\,\mathrm{d}\,\zeta\right\}\\ &= \operatorname{Im}\{f(i)\} - \operatorname{Re}\left\{\int_{\pi/3}^{\pi/2} \frac{e^{it}(1-e^{3it})^{1/3}}{1+e^{3it}}\,\mathrm{d}\,t\right\}\\ &> -2^{13/6}\int_{\pi/3}^{\pi/2} \frac{1}{\cos(3t/2)}\,\mathrm{d}\,t = +\infty.\end{aligned}$$

If f is starlike then for every $\zeta \in \mathbb{C}$ such that $0 < \operatorname{Re}\{\zeta\} < \operatorname{Re}\{f(e^{\pi/3})\}$ and $\operatorname{Im}\{\zeta\} > 0$ we have $\zeta \in f(\Delta)$. This implies that $f(i) \in f(\Delta)$ and so we have a contradiction.

Further possible examples of this type can be found in [3] and [4].

3. Coefficients estimates.

Theorem 3.1. *If $f \in COD_H$ satisfies* (1.1) *then for every $n = 2, 3, 4, \dots$,*

$$|a_n|^2 + |b_n|^2 \le (1 + |b_1|^2)n^2.$$

Proof. Suppose that $f \in CAD_H$. Since f is sense-preserving, $|b_1| < 1$. Hence both functions

$$\frac{h-g}{1-b_1} \quad \text{and} \quad \frac{h+g}{1+b_1}$$

belong to S by Theorem 2.4. Applying Theorem 2.1 to each of them we get for every $n = 2, 3, 4, \dots$,

$$\frac{|a_n - b_n|}{|1-b_1|} \le n \quad \text{and} \quad \frac{|a_n + b_n|}{|1+b_1|} \le n$$

and so

$$\begin{aligned}|a_n|^2 + |b_n|^2 &= \frac{1}{2}(|a_n - b_n|^2 + |a_n + b_n|^2)\\ &\le \frac{1}{2}(|1-b_1|^2 + |1+b_1|^2)n^2 = (1+|b_1|^2)n^2.\end{aligned}$$

Thus we have proved the theorem for $f \in CAD_H$. Now the theorem follows from Remark 1.1. □

Corollary 3.2. *If $f \in COD_H$ satisfies* (1.1) *then for every* $n = 2, 3, 4, \dots,$

$$|a_n| < \sqrt{2}n \quad \text{and} \quad |b_n| < \sqrt{2}n.$$

Proof. By Theorem 3.1 we have

$$|a_n| \le \sqrt{(1+|b_1|^2)n^2 - |b_n|^2} \le \sqrt{(1+|b_1|^2)}n,$$
$$|b_n| \le \sqrt{(1+|b_1|^2)n^2 - |a_n|^2} \le \sqrt{(1+|b_1|^2)}n.$$

Now, the corollary follows from the inequality $|b_1| < 1$. □

Corollary 3.3. *If $f \in COD_H^0$ satisfies* (1.1) *then for every* $n = 2, 3, 4, \dots,$

$$|a_n| \le n \quad \text{and} \quad |b_n| \le n.$$

Proof. Since $b_1 = 0$ for $f \in COD_H^0$ then from Theorem 3.1 we derive

$$|a_n| \le \sqrt{n^2 - |b_n|^2} \le n \quad \text{and} \quad |b_n| \le \sqrt{n^2 - |a_n|^2} \le n.$$

□

4. Distortion estimates.

Theorem 4.1. *If $f \in COD_H$ satisfies* (1.1) *then for every $z \in \Delta$,*

$$\frac{(1+|b_1|^2)(1-r)^2}{(1+r)^6} \le |h'(z)|^2 + |g'(z)|^2 \le \frac{(1+|b_1|^2)(1+r)^2}{(1-r)^6},$$

where $r := |z|$.

Proof. For $f \in CAD_H$ satisfying (1.1) both functions

$$\frac{h-g}{1-b_1} \quad \text{and} \quad \frac{h+g}{1+b_1}$$

belong to S by Theorem 2.4. By applying Theorem 2.2 to these functions we have

$$\frac{|1-b_1|^2(1-r)^2}{(1+r)^6} \le |h'(z) - g'(z)|^2 \le \frac{|1-b_1|^2(1+r)^2}{(1-r)^6} \tag{4.1}$$

and

$$\frac{|1+b_1|^2(1-r)^2}{(1+r)^6} \le |h'(z) + g'(z)|^2 \le \frac{|1+b_1|^2(1+r)^2}{(1-r)^6}. \tag{4.2}$$

Adding respective sides of (4.1) and (4.2) we prove the theorem for any $f \in CAD_H$. Then the theorem follows from Remark 1.1. □

Theorem 4.2. *If $f \in COD_H$ satisfies* (1.1) *then for every $z \in \Delta$,*

$$|h'(z)| \ge \begin{cases} \dfrac{(1-r)^{2\sqrt{2}-1}}{(1+r)^{2\sqrt{2}+1}}, & r \le r_0, \\ \dfrac{\sqrt{2}}{2}\dfrac{(1-r)}{(1+r)^3}, & r > r_0 \end{cases} \tag{4.3}$$

and

$$|h'(z)| \le \begin{cases} \dfrac{(1+r)^{2\sqrt{2}-1}}{(1-r)^{2\sqrt{2}+1}}, & r \le r_0, \\ \sqrt{2}\dfrac{(1+r)}{(1-r)^3}, & r > r_0 \end{cases} \tag{4.4}$$

and

$$0 \le |g'(z)| < \frac{(1+r)}{(1-r)^3}, \tag{4.5}$$

where $r := |z|$ *and* $r_0 := \left(2^{\frac{\sqrt{2}+1}{4}} - 1\right)\left(2^{\frac{\sqrt{2}+1}{4}} + 1\right)^{-1}$.

Proof. From Theorem 4.1 and the inequality $0 \le |b_1| < 1$ we have

$$\frac{(1-r)^2}{(1+r)^6} \le |h'(z)|^2 + |g'(z)|^2 < \frac{2(1+r)^2}{(1-r)^6}.$$

Since f is sense-preserving, we have $0 \le |g'(z)| < |h'(z)|$ for $z \in \Delta$. Hence, (4.5) and the following inequalities

$$\frac{\sqrt{2}}{2}\frac{(1-r)}{(1+r)^3} < |h'(z)| < \sqrt{2}\frac{(1+r)}{(1-r)^3} \tag{4.6}$$

hold.

We now prove that the estimate (4.6) can be improved for $r < r_0$. Fix $\zeta \in \Delta$ and $f \in COD_H$ satisfying (1.1). Applying disk automorphism $\Delta \ni z \mapsto (z+\zeta)(1+\overline{\zeta}z)^{-1}$ we see that the function

$$F(z) := \frac{f\left(\frac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)h'(\zeta)}$$

belongs to COD_H. Let $H(z) = z + A_2(\zeta)z^2 + A_3(\zeta)z^3 + A_4(\zeta)z^4 + \cdots$ be the analytic part of F. Then

$$A_2(\zeta) = \frac{1}{2}\left\{(1-|\zeta|^2)\frac{h''(\zeta)}{h'(\zeta)} - 2\overline{\zeta}\right\}.$$

By Corollary 3.2, $|A_2(\zeta)| \le 2\sqrt{2}$, which implies, after substitution $\zeta := z$, that

$$\frac{2r^2 - 4\sqrt{2}r}{1-r^2} \le \operatorname{Re}\left\{\frac{zh''(z)}{h'(z)}\right\} \le \frac{2r^2 + 4\sqrt{2}r}{1-r^2}, \quad z \in \Delta.$$

This inequality can be rewritten in the form

$$\frac{2r - 4\sqrt{2}}{1-r^2} \le \frac{\partial}{\partial r}\left\{\operatorname{Log}|h'(re^{i\theta})|\right\} \le \frac{2r + 4\sqrt{2}}{1-r^2}, \quad 0 \le r < 1, \tag{4.7}$$

where $z = re^{i\theta}$, $\theta \in \mathbb{R}$. Integrating each side of (4.7) we get the following estimate

$$\frac{(1-r)^{2\sqrt{2}-1}}{(1+r)^{2\sqrt{2}+1}} \le |h'(z)| \le \frac{(1+r)^{2\sqrt{2}-1}}{(1-r)^{2\sqrt{2}+1}}, \quad z \in \Delta. \tag{4.8}$$

Combining (4.6) with (4.8) we obtain

$$|h'(z)| \ge \max\left\{ \frac{\sqrt{2}}{2}\frac{(1-r)}{(1+r)^3}, \frac{(1-r)^{2\sqrt{2}-1}}{(1+r)^{2\sqrt{2}+1}} \right\} \tag{4.9}$$

and

$$|h'(z)| \le \min\left\{ \sqrt{2}\frac{(1+r)}{(1-r)^3}, \frac{(1+r)^{2\sqrt{2}-1}}{(1-r)^{2\sqrt{2}+1}} \right\}. \tag{4.10}$$

After simple calculations we derive from (4.9) and (4.10) the estimates (4.3) and (4.4), respectively. □

Corollary 4.3. *If $f \in COD_H^0$ satisfies* (1.1) *then for every $z \in \Delta$,*

$$\frac{(1-r)}{(1+r)^3\sqrt{1+r^2}} \le |h'(z)| \le \frac{(1+r)}{(1-r)^3} \tag{4.11}$$

and

$$0 \le |g'(z)| \le \frac{r(1+r)}{(1-r)^3\sqrt{1+r^2}}, \tag{4.12}$$

where $r := |z|$.

Proof. Since $b_1 = 0$ for $f \in COD_H^0$ then from Theorem 4.1 we derive

$$\frac{(1-r)^2}{(1+r)^6} \le |h'(z)|^2 + |g'(z)|^2 \le \frac{(1+r)^2}{(1-r)^6}. \tag{4.13}$$

The analytic dilatation $g'/\,h'$ of the function f satisfies the assumptions of Schwarz lemma, which yields

$$|g'(z)| \le |z||h'(z)|, \quad z \in \Delta.$$

Combining this inequality with (4.13) we obtain the estimates (4.11) and (4.12), which ends the proof. □

5. Growth estimates.

Theorem 5.1. *If $f \in COD_H$ satisfies* (1.1) *then for every $z \in \Delta$,*

$$\frac{(1+|b_1|^2)r^2}{(1+r)^4} \le |h(z)|^2 + |g(z)|^2 \le \frac{(1+|b_1|^2)r^2}{(1-r)^4},$$

where $r := |z|$.

Proof. For $f \in CAD_H$ satisfying (1.1) both functions

$$\frac{h-g}{1-b_1} \quad \text{and} \quad \frac{h+g}{1+b_1}$$

belong to S by Theorem 2.4. By applying Theorem 2.3 to these functions we have

$$\frac{|1-b_1|^2 r^2}{(1+r)^4} \le |h(z)-g(z)|^2 \le \frac{|1-b_1|^2 r^2}{(1-r)^4} \tag{5.1}$$

and

$$\frac{|1+b_1|^2 r^2}{(1+r)^4} \le |h(z)+g(z)|^2 \le \frac{|1+b_1|^2 r^2}{(1-r)^4}. \tag{5.2}$$

Adding respective sides of (5.1) and (5.2) we prove the theorem for any $f \in CAD_H$. Then the theorem follows from Remark 1.1. □

Corollary 5.2. *If $f \in COD_H$ satisfies (1.1) then for every $z \in \Delta$,*

$$0 \le |h(z)| < \sqrt{2}\frac{r}{(1-r)^2} = \frac{\sqrt{2}}{4}\left[\left(\frac{1+r}{1-r}\right)^2 - 1\right] \tag{5.3}$$

and

$$0 \le |g(z)| < \sqrt{2}\frac{r}{(1-r)^2} = \frac{\sqrt{2}}{4}\left[\left(\frac{1+r}{1-r}\right)^2 - 1\right] \tag{5.4}$$

and

$$0 \le |f(z)| < 2\frac{r}{(1-r)^2} = \frac{1}{2}\left[\left(\frac{1+r}{1-r}\right)^2 - 1\right], \tag{5.5}$$

where $r := |z|$.

Proof. From Theorem 5.1 and the inequality $0 \le |b_1| < 1$ we get

$$\frac{r^2}{(1+r)^4} \le |h(z)|^2 + |g(z)|^2 < \frac{2r^2}{(1-r)^4}. \tag{5.6}$$

The estimates (5.3) and (5.4) follow from (5.6) and the trivial inequalities $|h(z)| \ge 0$ and $|g(z)| \ge 0$, respectively. The estimate (5.5) we derive from (5.6) and the following inequality

$$|f(z)| = \left|h(z) + \overline{g(z)}\right| \le |h(z)| + |g(z)| \le \sqrt{2(|h(z)|^2 + |g(z)|^2)}. \tag{5.7}$$

□

The estimates (5.3) and (5.4) given in Corollary 5.2 can be improved. Moreover we can obtain a lower estimate for $|h(z)|$.

Theorem 5.3. *If $f \in COD_H$ satisfies* (1.1) *then for every $z \in \Delta$,*

$$|h(z)| \geq \begin{cases} \frac{\sqrt{2}}{8}\left[1-\left(\frac{1-r}{1+r}\right)^{2\sqrt{2}}\right], & r \leq r_0, \\ 2^{\frac{-\sqrt{2}-7}{2}}\left(\sqrt{2}-1\right)+\frac{\sqrt{2}}{8}\left[1-\left(\frac{1-r}{1+r}\right)^{2}\right], & r > r_0 \end{cases} \tag{5.8}$$

and

$$|h(z)| \leq \begin{cases} \frac{\sqrt{2}}{8}\left[\left(\frac{1+r}{1-r}\right)^{2\sqrt{2}}-1\right], & r \leq r_0, \\ \frac{\sqrt{2}}{8}\left[1-2^{\frac{\sqrt{2}+2}{2}}\left(\sqrt{2}-1\right)\right]+\frac{\sqrt{2}}{4}\left[\left(\frac{1+r}{1-r}\right)^{2}-1\right], & r > r_0 \end{cases} \tag{5.9}$$

and

$$0 \leq |g(z)| < \frac{1}{4}\left[\left(\frac{1+r}{1-r}\right)^{2}-1\right], \tag{5.10}$$

where $r := |z|$ and $r_0 := \left(2^{\frac{\sqrt{2}+1}{4}}-1\right)\left(2^{\frac{\sqrt{2}+1}{4}}+1\right)^{-1}$.

Proof. The proof is based on Theorem 4.2. Fix $f \in COD_H$ satisfying (1.1). First we prove the estimate (5.8) for every $z \in \Delta(0,R)$, where $R \in (0,1]$ is some constant and $\Delta(0,r) := \{z \in \mathbb{C} : |z| < r\}$, under the following additional assumption

$$h(z) \neq 0, \quad z \in \Delta(0,R) \setminus \{0\}. \tag{5.11}$$

Fix $r \in (0,R)$. By the normalization we have $h(0)=0$, which implies that $0 \in h(\Delta(0,r))$. Hence there exists $z \in \mathrm{T}(0,r)$, where $\mathrm{T}(0,r) := \{z \in \mathbb{C} : |z| = r\}$ such that $|h(z)| = \min_{\zeta \in \mathrm{T}(0,r)} |h(\zeta)| > 0$ and $[0,h(z)] \subset h(\overline{\Delta}(0,r))$, where $\overline{\Delta}(0,r) := \Delta(0,r) \cup \mathrm{T}(0,r)$. Now observe that h is locally univalent in Δ. Therefore $\Gamma := h^{-1}([0,h(z)])$ is a Jordan arc and h is univalent on Γ. Applying the estimate (4.3) we get

$$\begin{aligned} |h(z)| &= \int_\Gamma |h'(\zeta)||\,\mathrm{d}\,\zeta| \\ &\geq \begin{cases} \int_0^r \frac{(1-\rho)^{2\sqrt{2}-1}}{(1+\rho)^{2\sqrt{2}+1}}\,\mathrm{d}\,\rho, & r \leq r_0, \\ \int_0^{r_0} \frac{(1-\rho)^{2\sqrt{2}-1}}{(1+\rho)^{2\sqrt{2}+1}}\,\mathrm{d}\,\rho + \int_{r_0}^r \frac{\sqrt{2}}{2}\frac{(1-\rho)}{(1+\rho)^3}\,\mathrm{d}\,\rho, & r > r_0. \end{cases} \end{aligned} \tag{5.12}$$

After calculations we derive from (5.12) the estimate (5.8) for every $z \in \Delta(0,R)$. It remains to show that (5.11) holds for $R=1$. Since h is analytic in Δ, then we know that the set $A := \{z \in \Delta : h(z)=0\}$ cannot contain any sequence converging to 0. Obviously, $0 \in A$. Assume that $A \neq \{0\}$. Thus we can choose $A \ni \hat{z} \neq 0$ such that $|\hat{z}| = \hat{r}$, where $\hat{r} := \min_{\zeta \in A\setminus\{0\}} |\zeta|$. Observe that $\hat{r} \in (0,1)$ and (5.11) holds for $R = \hat{r}$. Hence for every $z \in \Delta(0,\hat{r})$ we have the estimate (5.8) from which we obtain that for every $z \in \Delta(0,\hat{r})$, $|h(z)|$ is bounded away from 0. On the other hand h is a continuous function in Δ, and so for any sequence $\{z_n\}$ of $z_n \in \Delta(0,\hat{r})$, $n = 1,2,3,\ldots$, such that $z_n \to \hat{z}$ we have $h(z_n) \to h(\hat{z}) = 0$. Thus we get a contradiction, which implies (5.11) is valid for $R = 1$ and completes the proof of (5.8) for every $z \in \Delta$.

Let $\gamma := [0,z]$. Applying the estimate (4.4) we have

$$|h(z)| = \left|\int_\gamma h'(\zeta)\,\mathrm{d}\,\zeta\right| \leq \int_\gamma |h'(\zeta)||\,\mathrm{d}\,\zeta|$$

$$= \begin{cases} \displaystyle\int_0^r \frac{(1+\rho)^{2\sqrt{2}-1}}{(1-\rho)^{2\sqrt{2}+1}}\,\mathrm{d}\,\rho, & r \leq r_0, \\ \displaystyle\int_0^{r_0} \frac{(1+\rho)^{2\sqrt{2}-1}}{(1-\rho)^{2\sqrt{2}+1}}\,\mathrm{d}\,\rho + \int_{r_0}^r \sqrt{2}\frac{(1+\rho)}{(1-\rho)^3}\,\mathrm{d}\,\rho, & r > r_0. \end{cases} \tag{5.13}$$

Simplifying (5.13) we derive (5.9). The estimate (5.10) follows in a similar way to the proof of the estimate (5.9). □

Corollary 5.4. *If $f \in COD_H^0$ satisfies* (1.1) *then for every $z \in \Delta$,*

$$0 \leq |h(z)| \leq \frac{r}{(1-r)^2} \tag{5.14}$$

and

$$0 \leq |g(z)| \leq \frac{r}{(1-r)^2} \tag{5.15}$$

and

$$0 \leq |f(z)| \leq \sqrt{2}\frac{r}{(1-r)^2}, \tag{5.16}$$

where $r := |z|$.

Proof. Since $b_1 = 0$ for $f \in COD_H^0$ then from Theorem 5.1 we derive

$$\frac{r^2}{(1+r)^4} \leq |h(z)|^2 + |g(z)|^2 \leq \frac{r^2}{(1-r)^4}. \tag{5.17}$$

The estimates (5.14) and (5.15) follow from (5.17) and the trivial inequalities $|h(z)| \geq 0$ and $|g(z)| \geq 0$, respectively. The estimate (5.16) we derive from (5.17) and the inequality (5.7). □

The estimate (5.15) given in Corollary 5.4 can be improved and a lower estimate for $|h(z)|$ can be obtained by the same method as in the proof of Theorem 5.3.

Theorem 5.5. *If $f \in COD_H^0$ satisfies* (1.1) *then for every $z \in \Delta$,*

$$|h(z)| \geq \frac{3}{4} - \frac{(3+r)\sqrt{1+r^2}}{4(1+r)^2} + \frac{\sqrt{2}}{8} \operatorname{Log} \frac{1-r+\sqrt{2}\sqrt{1+r^2}}{(1+\sqrt{2})(1+r)} \tag{5.18}$$

and

$$0 \leq |g(z)| \leq \frac{1}{4} - \frac{(1-3r)\sqrt{1+r^2}}{4(1-r)^2} + \frac{\sqrt{2}}{8} \operatorname{Log} \frac{(1+\sqrt{2})(1-r)}{1+r+\sqrt{2}\sqrt{1+r^2}}, \tag{5.19}$$

where $r := |z|$.

Proof. The proof of the theorem is based on Corollary 4.3. Fix $f \in COD_H^0$ satisfying (1.1). Let $\Gamma := h^{-1}([0, h(z)])$. Applying the estimate (4.11) we have

$$|h(z)| = \int_\Gamma |h'(\zeta)||\,\mathrm{d}\,\zeta| \geq \int_0^r \frac{(1-\rho)}{(1+\rho)^3\sqrt{1+\rho^2}}\,\mathrm{d}\,\rho.$$

After simplifying we get (5.18). Let $\gamma := [0, z]$. Applying (4.12) we get

$$|g(z)| = \left|\int_\gamma g'(\zeta)\,\mathrm{d}\,\zeta\right| \leq \int_\gamma |g'(\zeta)||\,\mathrm{d}\,\zeta| \leq \int_0^r \frac{\rho(1+\rho)}{(1-\rho)^3\sqrt{1+\rho^2}}\,\mathrm{d}\,\rho.$$

Again simplifying we get (5.19). □

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References

[1] Clunie, J. G., Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **9** (1984), 3–25.

[2] de Branges, L., *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.

[3] Dorff, M., Szynal, J., *Harmonic shears of elliptic integrals*, Rocky Mountain J. Math. **35** (2005), 485–499.

[4] Driver, K., Duren, P. L., *Harmonic shears of regular polygons by hypergeometric functions*, J. Math. Anal. Appl. **239** (1999), 72–84.

[5] Duren, P. L., *Univalent Functions*, Grundlehren der matematischen Wissenschaften 259, Springer-Verlag, New York, 1983.

[6] Duren, P. L., *Harmonic Mappings in the Plane*, Cambridge tracts in mathematics 156, Cambridge Univ. Press, Cambridge, 2004.

[7] Goodman, A. W., *Univalent Functions*, Mariner Publ. Co. Inc., Tampa, 1983.

[8] Nehari, Z., *Conformal Mapping*, Dover Publ. Inc., New York, 1975.

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