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On pseudo projectively flat LP-Sasakian manifold with a coefficient α

Abstract. Recently, the notion of Lorentzian almost paracontact manifolds with a coefficient α has been introduced and studied by De et al. [1]. In the present paper we investigate pseudo projectively flat LP-Sasakian manifold with a coefficient α.

1. Introduction. In 1989, Matsumoto [2] introduced the notion of LP-Sasakian manifolds. Then Mihai and Rosca [3] introduced the same notion independently and they obtained several results in this manifold. In a recent paper, De, Shaikh and Sengupta [1] introduced the notion of LP-Sasakian manifolds with a coefficient α, which generalizes the notion of LP-Sasakian manifolds.

In the present paper we study pseudo projectively flat LP-Sasakian manifold with a coefficient α. Here we prove that in a pseudo projectively flat LP-Sasakian manifolds with a coefficient α the characteristic vector field is a concircular vector field if and only if the manifold is η-Einstein and pseudo projectively flat LP-Sasakian manifold with a coefficient α is a manifold of constant curvature if the scalar curvature r is a constant.

2. Preliminaries. Let M be the n-dimensional differential manifold endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant

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vector field η and a Lorentzian metric g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-,+,+,\dots,+)$, where T_pM denotes the tangent vector space of M at p and R is the real number space, which satisfies

$$\eta(\xi) = -1, \qquad \phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$g(X,\xi) = \eta(X), \qquad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X and Y. Then such a structure (ϕ,ξ,η,g) is termed as Lorentzian almost paracontact structure and the manifold M with the structure (ϕ,ξ,η,g) is called Lorentzian almost paracontact manifold M [2]. In the Lorentzian almost paracontact manifold M, the following relations hold [2]:

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \tag{2.3}$$

$$\omega(X,Y) = \omega(Y,X) \tag{2.4}$$

where $\omega(X,Y) = g(X,\phi Y)$. In the Lorentzian almost paracontact manifold M, if the relations

$$\begin{aligned}(\nabla_Z\omega)(X,Y) = \alpha[&(g(X,Z) + \eta(X)\eta(Z))\eta(Y) \\ &+ (g(Y,Z) + \eta(Y)\eta(Z))\eta(X)]\end{aligned} \tag{2.5}$$

and

$$\omega(X,Y) = \frac{1}{\alpha}(\nabla_X\eta)(Y) \tag{2.6}$$

hold, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g, then M is called an LP-Sasakian manifold with a coefficient α [1]. An LP-Sasakian manifold with coefficient 1 is an LP-Sasakian manifold [2].

If a vector field V satisfies the equation of the following form:

$$\nabla_X V = \beta X + T(X)V,$$

where β is a non-zero scalar function and T is a covariant vector field, then V is called a torse-forming vector field [5].

In a Lorentzian manifold M, if we assume that ξ is a unit torse-forming vector field, then

$$(\nabla_X\eta)(Y) = \alpha[g(X,Y) + \eta(X)\eta(Y)], \tag{2.7}$$

where α is a non-zero scalar function. Hence the manifold admitting a unit torse-forming vector field satisfying (2.7) is an LP-Sasakian manifold with a coefficient α. And, if η satisfies

$$(\nabla_X\eta)(Y) = \varepsilon[g(X,Y) + \eta(X)\eta(Y)], \qquad \varepsilon^2 = 1, \tag{2.8}$$

then M is called an LSP-Sasakian manifold [2]. In particular, if α satisfies (2.7) and the equation of the following form:

$$\alpha(X) = P\eta(X), \qquad \alpha(X) = \nabla_X \alpha, \tag{2.9}$$

where P is a scalar function, then ξ is called a concircular vector field.

Let us consider an LP-Sasakian manifold M with the structure (ϕ, ξ, η, g) and with a coefficient α. Then we have the following relations [1]:

$$\begin{aligned}\eta(R(X,Y)Z) = &-\alpha(X)\omega(Y,Z) + \alpha(Y)\omega(X,Z) \\ &+ \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\end{aligned} \tag{2.10}$$

and

$$S(X,\xi) = -\psi\alpha(X) + (n-1)\alpha^2\eta(X) + \alpha(\phi X), \tag{2.11}$$

where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold and $\psi = Trace\,(\phi)$.

We now state the following results, which are used in the later section.

Lemma 2.1 ([1])**.** *In an LP-Sasakian manifold M with a non-constant coefficient α, one of the following cases occurs:*

i) $\psi^2 = (n-1)^2$
ii) $\alpha(Y) = -P\eta(Y)$,

where $P = \alpha(\xi)$.

Lemma 2.2 ([1])**.** *In a Lorentzian almost paracontact manifold $M(\phi, \xi, \eta, g)$ with its structure (ϕ, ξ, η, g) satisfying $\omega(X,Y) = \frac{1}{\alpha}(\nabla_X \eta)(Y)$, where α is a non-zero scalar function, the vector field ξ is torse-forming if and only if the relation $\psi^2 = (n-1)^2$ holds.*

3. Pseudo projectively flat LP-Sasakian manifold with a coefficient $\boldsymbol{\alpha}$. Let us consider a pseudo projectively flat LP-Sasakian manifold M $(n > 3)$ with a coefficient α. First suppose that α is not constant. Then since the pseudo projective curvature tensor vanishes, the curvature tensor $'R$ satisfies [4]

$$\begin{aligned}'R(X,Y,Z,W) = &-\frac{b}{a}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)] \\ &+ \frac{r}{n}\left[\frac{1}{n-1} + \frac{b}{a}\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned} \tag{3.1}$$

and

$$'R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

where a, b are constants such that $a, b \neq 0$ and $a + b(n-1) \neq 0$, r is the scalar curvature of the manifold. Putting $W = \xi$ in (3.1) and then using

(2.10) and (2.11), we get

$$\begin{aligned}
&-\alpha(X)\omega(Y,Z)+\alpha(Y)\omega(X,Z)+\alpha^2[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\\
&=-\frac{b}{a}[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)]\\
&+\frac{r}{n}\left[\frac{1}{n-1}+\frac{b}{a}\right][g(Y,Z)\eta(X)-g(X,Z)\eta(Y)].
\end{aligned}\tag{3.2}$$

Again if we put $X=\xi$ in (3.2) and using (2.3) and (2.11), we obtain

$$\begin{aligned}
S(Y,Z)&=\left[-\frac{a}{b}\alpha^2+\frac{ar}{bn(n-1)}+\frac{r}{n}\right]g(Y,Z)\\
&+\left[-\frac{a}{b}\alpha^2-(n-1)\alpha^2+\frac{ar}{bn(n-1)}+\frac{r}{n}\right]\eta(Y)\eta(Z)\\
&+\psi\alpha(Z)-\alpha(\phi Z)\eta(Y)-\frac{a}{b}P\omega(Y,Z)
\end{aligned}\tag{3.3}$$

where $P=\alpha(\xi)$.

If an LP-Sasakian manifold M with the coefficient α satisfies the relation

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),$$

where a, b are the associated functions on the manifold, then the manifold M is called an η-Einstein manifold. Then we have [1]

$$\begin{aligned}
S(X,Y)&=\left[\frac{r}{n-1}-\alpha^2-\frac{P\psi}{n-1}\right]g(X,Y)\\
&+\left[\frac{r}{n-1}-n\alpha^2-\frac{nP\psi}{n-1}\right]\eta(X)\eta(Y).
\end{aligned}\tag{3.4}$$

Putting $X=Y=e_i$, in (3.4), where $\{e_i\}$ is an orthonormal basis of the tangent space at a point of the manifold and taking summation over $1\leq i\leq n$, we get

$$r=n(n-1)\alpha^2+n\psi P.\tag{3.5}$$

By virtue of (3.3) and (3.4) we get

$$\begin{aligned}
&\left[\frac{\alpha^2}{b}(a-b)+\frac{r(b-a)}{n(n-1)b}-\frac{P\psi}{(n-1)}\right]g(Y,Z)-\psi\alpha(Z)-\alpha(\phi Z)\eta(Y)\\
&+\left[\frac{\alpha^2}{b}(a-b)+\frac{r(b-a)}{n(n-1)b}-\frac{nP\psi}{(n-1)}\right]\eta(Y)\eta(Z)\\
&+\frac{a}{b}P\omega(Y,Z)=0.
\end{aligned}\tag{3.6}$$

Putting $Y=\xi$ in (3.6), we obtain

$$\psi\alpha(Z)-\alpha(\phi Z)=-\psi P\eta(Z),$$

for all Z. Replace Z by Y in the above equation, we get

$$\psi\alpha(Y) - \alpha(\phi Y) = -\psi P\eta(Y), \tag{3.7}$$

for all Y. Using (3.7) in (3.6) and then by virtue of (3.5) we get

$$P\frac{a}{b}\left[\frac{\psi}{n-1}[g(Y,Z) + \eta(Y)\eta(Z)] + \omega(Y,Z)\right] = 0. \tag{3.8}$$

If $P = 0$, then from (3.7) we have $\alpha(\phi Y) = \psi\alpha(Y)$. Thus ψ is equal to ± 1 as ψ is an eigenvalue of the matrix (ϕ). Hence, by virtue of Lemma 2.1, we get $\alpha(Y) = 0$ for all Y and so α is constant, which contradicts our assumption.

Consequently, we have $P \neq 0$ and hence from (3.8) we get

$$\frac{a}{b}\left[\frac{\psi}{n-1}[g(Y,Z) + \eta(Y)\eta(Z)] + \omega(Y,Z)\right] = 0. \tag{3.9}$$

Putting $Y = \phi Y$ in (3.9) and then using (2.3), we obtain

$$\frac{a}{b}\left[\frac{\psi}{n-1}\omega(Y,Z) + [g(Y,Z) + \eta(Y)\eta(Z)]\right] = 0. \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$\{\psi^2 - (n-1)^2\}[g(Y,Z) + \eta(Y)\eta(Z)] = 0,$$

which gives by virtue of $n > 1$

$$\psi^2 = (n-1)^2. \tag{3.11}$$

Hence Lemma 2.2 proves that ξ is torse-forming.

We have

$$(\nabla_X\eta)(Y) = \beta\{g(X,Y) + \eta(X)\eta(Y)\}.$$

Then from (2.6) we get

$$\omega(X,Y) = \frac{\beta}{\alpha}\{g(X,Y) + \eta(X)\eta(Y)\} = g\left(\frac{\beta}{\alpha}(X + \eta(X)\xi), Y\right)$$

and $\omega(X,Y) = g(\phi X, Y)$.

Since g is non-singular, we have

$$\phi(X) = \frac{\beta}{\alpha}(X + \eta(X)\xi)$$

and

$$\phi^2(X) = \left(\frac{\beta}{\alpha}\right)^2(X + \eta(X)\xi).$$

It follows from (2.1) that $\left(\frac{\beta}{\alpha}\right)^2 = 1$ and hence, $\alpha = \pm\beta$. Thus we have

$$\phi(X) = \pm(X + \eta(X)\xi).$$

By virtue of (3.7) we see that $\alpha(Y) = -P\eta(Y)$, where $P = \alpha(\xi)$. Thus, we conclude that ξ is a concircular vector field. Conversely, we suppose that

ξ is a concircular vector field. Then we have the equation of the following form:

$$(\nabla_X \eta)(Y) = \beta\{g(X,Y) + \eta(X)\eta(Y)\},$$

where β is a certain function and $\nabla_X \beta = q\eta(X)$ for a certain scalar function q. Hence by virtue of (2.6) we have $\alpha = \pm\beta$. Thus

$$\Omega(X,Y) = \varepsilon\{g(X,Y) + \eta(X)\eta(Y)\}, \qquad \varepsilon^2 = 1,$$

$$\psi = \varepsilon(n-1), \qquad \nabla_X \alpha = \alpha(X) = p\eta(X), \qquad p = \varepsilon q.$$

Using these relations in (3.3) and (3.7), it can be easily seen that M is η-Einstein. Thus we can state the following:

Theorem 3.1. *In a pseudo projectively flat LP-Sasakian manifold M $(n > 1)$ with a non-constant coefficient α, the characteristic vector field ξ is a concircular vector field if and only if M is η-Einstein.*

Next we consider the case where the coefficient α is constant. In this case the following relations hold:

$$\eta(R(X,Y)Z) = \alpha^2\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{3.12}$$

$$S(X,\xi) = (n-1)\alpha^2\eta(X). \tag{3.13}$$

Putting $W = \xi$ in (3.1) and then using (3.12) and (3.13), we get

$$\begin{aligned} &a \cdot \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + b[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] \\ &- \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] = 0. \end{aligned} \tag{3.14}$$

Again putting $X = \xi$ in (3.14) we get by virtue of (3.13) that

$$\begin{aligned} S(Y,Z) = &\left[\frac{r}{n}\left(1 + \frac{a}{b(n-1)}\right) - \frac{a}{b}\alpha^2\right] g(Y,Z) \\ &+ \frac{(a + b(n-1))}{b}\left[\frac{r}{n(n-1)} - \alpha^2\right]\eta(Y)\eta(Z) \end{aligned} \tag{3.15}$$

Hence we can state the following:

Theorem 3.2. *A pseudo projectively flat LP-Sasakian manifold M $(n > 1)$ with a constant coefficient α is an η-Einstein manifold.*

Differentiating (3.15) covariantly along X and making use of (2.6) we get

$$\begin{aligned} (\nabla_X S)(Y,Z) = &\frac{dr(X)}{n-1}\left(1 + \frac{a}{b(n-1)}\right)[g(Y,Z) + \eta(Y)\eta(Z)] \\ &+ \frac{\alpha(a + b(n-1))}{b}\left[\frac{r}{n(n-1)} - \alpha^2\right] \\ &\times [\omega(X,Y)\eta(Z) + \omega(X,Z)\eta(Y)] \end{aligned}$$

where $dr(X) = \nabla_X r$. This implies that

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)$$

$$\begin{aligned} &= \frac{dr(X)}{n-1}\left(1+\frac{a}{b(n-1)}\right)[g(Y,Z)+\eta(Y)\eta(Z)] \\ &- \frac{dr(Y)}{n-1}\left(1+\frac{a}{b(n-1)}\right)[g(X,Z)+\eta(X)\eta(Z)] \\ &+ \frac{\alpha(a+b(n-1))}{b}\left[\frac{r}{n(n-1)}-\alpha^2\right] \\ &\times [\omega(X,Z)\eta(Y)-\omega(Y,Z)\eta(X)]. \end{aligned} \tag{3.16}$$

On the other hand, in our case, since we have $(\nabla_X \overline{P})(X,Y)Z = 0$, we get $div\overline{P} = 0$, where "div" denotes the divergence. So for $n > 1$, $div\overline{P} = 0$ gives

$$\begin{aligned} &(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &= \frac{1}{n(a+b)}\left[\frac{a+(n-1)b}{n-1}\right][g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \end{aligned} \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\begin{aligned} &\frac{1}{n(a+b)}\left[\frac{a+(n-1)b}{n-1}\right][g(Y,Z)dr(X) - g(X,Z)dr(Y)] \\ &= \frac{dr(X)}{n-1}\left(1+\frac{a}{b(n-1)}\right)[g(Y,Z)+\eta(Y)\eta(Z)] \\ &+ \frac{dr(Y)}{n-1}\left(1+\frac{a}{b(n-1)}\right)[g(X,Z)+\eta(X)\eta(Z)] \\ &+ \frac{\alpha(a+b(n-1))}{b}\left[\frac{r}{n(n-1)}-\alpha^2\right] \\ &\times [\omega(X,Z)\eta(Y)+\omega(Y,Z)\eta(X)]. \end{aligned} \tag{3.18}$$

If r is constant, then from (3.18) we obtain

$$\frac{\alpha(a+b(n-1))}{b}\left[\frac{r}{n(n-1)}-\alpha^2\right] = 0.$$

Since $a + b(n-1) \neq 0$, the above equation gives

$$r = n(n-1)\alpha^2. \tag{3.19}$$

Now substituting (3.15) in (3.1) we get

$$\begin{aligned} {}'R(X,Y,Z,W) &= \alpha^2[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ \left[\frac{(a+b(n-1))}{a}\left(\frac{r}{n(n-1)}-\alpha^2\right)\right] \\ &\times [g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)]. \end{aligned} \tag{3.20}$$

Hence by using (3.19) in (3.20) it follows that,

$$'R(X,Y,Z,W) = \alpha^2[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

This shows that the manifold is of constant curvature. Thus we can state the following:

Theorem 3.3. *In a pseudo projectively flat LP-Sasakian manifold M $(n > 1)$ with a constant coefficient α, if the scalar curvature r is constant, then M is of constant curvature.*

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