ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXI, 2007 SECTIO A 9–14

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On convexity of the space of random elements

Abstract. In the space of random elements taking values in a metric space convex in the sense of Doss we may define expected value (see [2]). In this paper we show that the space of random elements with a proper metric is also convex in the sense of Doss if the space of values is convex in the sense of Doss.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. By (S, ϱ) we denote a metric space and ζ stands for the σ-field generated by the open sets of S. Throughout this note S is assumed to be a nondegenerate, separable and complete space. A mapping $X\colon \Omega \to S$ such that $X^{-1}(\zeta) \subset \mathcal{A}$, is called a random element (r.e.). The set of all r.e. is denoted by $\mathcal{X}_S$. On this set we may introduce the following well-known metrics:
Ky Fan metric

$$r(X,Y) = \inf\{\varepsilon > 0 : P(\varrho(X,Y) > \varepsilon) < \varepsilon\}$$

and

$$r_1(X,Y) = E\frac{\varrho(X,Y)}{1+\varrho(X,Y)}.$$

The convergence in both metrics is equivalent to each other and to the convergence in probability [1].

2000 *Mathematics Subject Classification.* 60B99, 60A10, 28A99.
Key words and phrases. Doss expectation, convexity, metric space.

In the literature there are many different definitions of convexity in metric spaces namely:
Convexity in the sense of Menger

$$\forall_{x_1,x_2\in S}\ \exists_{t\in S,\ t\neq x_1,x_2}\quad \varrho(x_1,t)+\varrho(t,x_2)=\varrho(x_1,x_2).$$

Convexity

$$\forall_{x_1,x_2\in S}\ \forall_{p\in[0,1]}\ \exists_{t\in S}\quad \varrho(x_1,t)=p\varrho(x_1,x_2),\ \ \varrho(t,x_2)=(1-p)\varrho(x_1,x_2).$$

Strict convexity

$$\forall_{x_1,x_2\in S}\ \forall_{p\in[0,1]}\ \exists_{t\in S}\quad \varrho(x_1,t)=p\varrho(x_1,x_2),\ \ \varrho(t,x_2)=(1-p)\varrho(x_1,x_2)$$

and element t is uniquely determined.
Convexity in the sense of Doss

$$\forall_{x_1,x_2\in S}\ \exists_{t\in S}\ \forall_{z\in S}\quad \varrho(z,t)\leq\frac{1}{2}\left(\varrho(x_1,z)+\varrho(x_2,z)\right). \tag{1}$$

W. Zięba in [4] shown that if a separable, complete metric space (S,ϱ) is convex then $\mathcal{X}$ with the Ky Fan metric is convex, and also a set of probabilistic measures $\mathcal{P}(S)$ with the Levy–Prokhorov metric is a convex metric space. It is quite obvious that if a metric space is separable and complete then convexity in the sense of Menger is equivalent to the convexity. The following examples show that if a metric space (S,ϱ) has any of other listed geometrical properties then the space of random elements with a given metric equivalent to convergence in probability may not share this property.

Example 1. Let $S=\mathbb{R}$ with $|\cdot|$ metric. Let us check convexity in the sense of Doss. Let $P(X_1=-1)=1$ and $P(X_2=1)=1$. Then $r(X_1,X_2)=1$. We will show that there is no such random element T that

$$\forall_{Z\in\mathcal{X}_S}\quad r(Z,T)\leq\frac{1}{2}\left(r(X_1,Z)+r(X_2,Z)\right). \tag{2}$$

Suppose that such an element exists. Taking in (2) $Z=X_1$ and $Z=X_2$ we get respectively

$$r(X_1,T)\leq\frac{1}{2}r(X_1,X_2)=\frac{1}{2},\qquad r(X_2,T)\leq\frac{1}{2}r(X_1,X_2)=\frac{1}{2}.$$

Since $r(\cdot,\cdot)$ is a metric we have

$$r(X_1,T)=r(X_2,T)=\frac{1}{2}.$$

According to the definition of the Ky Fan metric it implies that

$$P\left(T\in\left[-\frac{3}{2},-\frac{1}{2}\right]\right)\geq\frac{1}{2},\qquad P\left(T\in\left[\frac{1}{2},\frac{3}{2}\right]\right)\geq\frac{1}{2}.$$

Now we define $T(\omega) \in \left[-\frac{3}{2}, -\frac{1}{2}\right]$ for $\omega \in A$ and $T(\omega) \in \left[\frac{1}{2}, \frac{3}{2}\right]$ for $\omega \in A'$, where $P(A) = P(A') = \frac{1}{2}$. We take

$$Z(\omega) = \begin{cases} 1, & \omega \in A; \\ -1, & \omega \in A'. \end{cases} \tag{3}$$

Then, we have $r(Z, X_1) = r(Z, X_2) = \frac{1}{2}$ and $r(Z, T) = 1$ which contradicts (2).

Remark 1. This example also shows that strict convexity also is not necessarily shared with spaces of random elements or probability measures. In fact it is easy to see that T and Z are two different "midpoints" of "segment" (X_1, X_2).

Remark 2. Note that the Ky Fan metric takes value 1 always when values of random elements X, Y satisfy the condition

$$\varrho\left(X(\omega), Y(\omega)\right) \geq 1 \quad a.s. \tag{4}$$

In the view of the last remark the following questions appear:
Maybe the Ky Fan metric is "wrong", maybe in other metric we can get the property we need? Maybe if the space (S, ϱ) has diameter less or equal to 1 $(\forall (x, y \in S)\ \varrho(x, y) \leq 1)$ we can obtain property we need?

The partial answer to these questions is given by the following.

Example 2. Let $S = [0, 1]$ with $|\cdot|$ metric. Let $P\left(X_1 = 0\right) = 1$, $P\left(X_2 = 1\right) = 1$. Then $r_1(X_1, X_2) = \frac{1}{2}$. Suppose that such element exists. Using analogous procedure we have $r_1(X_1, T) = r_1(X_2, T) = \frac{1}{4}$ so, since we have strict inequality $\frac{x}{1+x} < x$ for $x \in (0, 1]$

$$P\left(T = 0\right) = \frac{1}{2}, \quad P\left(T = 1\right) = \frac{1}{2}$$

(here, this is a uniquely (in distribution) determined midpoint of the segment (X_1, X_2)). Now suppose that $T(\omega) = 0$ for $\omega \in A$ and $T(\omega) = 1$ for $\omega \in A'$ where $P(A) = \frac{1}{2}$. Set

$$Z(\omega) = \begin{cases} 1, & \omega \in A; \\ 0, & \omega \in A'; \end{cases} \tag{5}$$

now we have

$$r_1(Z, X_1) = r_1(Z, X_2) = \frac{1}{4}, \quad r_1(Z, T) = \frac{1}{2}$$

which contradicts (2).

Now there is the following question:
Is there a metric d on $\mathcal{X}$ convergence in which would be equivalent to the convergence in probability and such that $(\mathcal{X}, d)$ would be convex in the sense of Doss?

In some cases we may find such metric.

Theorem 1. *Let (S,ϱ) be a separable, complete metric space satisfying condition (1) and such that*

$$\exists_{M\in\mathbb{R}} \sup_{x,y\in S} \varrho(x,y) < M. \tag{6}$$

Then $(\mathcal{X}_S, d)$, where $d(X,Y) = E\varrho(X,Y)$, satisfies (2) and convergence in probability is equivalent to the convergence in metric.

To prove this theorem we will need the following lemma.

Lemma 1. *Let X_1, X_2 be r.e. defined on $(\Omega,\mathcal{A},P)$ with values in the separable, complete metric space (S,ϱ). If (S,ϱ) satisfies (1) then there exists a r.e. T such that*

$$\forall_{z\in S}\ \forall_{\omega\in\Omega} \quad \varrho\left(z,T(\omega)\right) \le \frac{1}{2}\left[\varrho\left(z,X_1(\omega)\right) + \varrho\left(z,X_2(\omega)\right)\right].$$

Proof of Lemma 1. We choose a sequence of Borel subsets $S_{i_1,i_2,\ldots,i_k}$ satisfying the following conditions [3]:

(1) $S_{i_1,i_2,\ldots,i_k} \cap S_{i'_1,i'_2,\ldots,i'_k} = \emptyset$ if $i_s \neq i'_s$ for some $1 \le s \le k$,

(2) $\bigcup_{i_k=1}^{\infty} S_{i_1,i_2,\ldots,i_{k-1},i_k} = S_{i_1,i_2,\ldots,i_{k-1}}$, $\bigcup_{i_1=1}^{\infty} S_{i_1} = S$,

(3) $\sup_{x,y\in S_{i_1,i_2,\ldots,i_k}} \varrho(x,y) \le \frac{1}{2^k}$.

Let $W = \{w_1, w_2, \ldots\}$ be a dense subset in S. Define

$$A_{i_1,i_2,\ldots,i_k} = \bigcap_{j=1}^{\infty}\left\{\omega : \inf_{x\in S_{i_1,i_2,\ldots,i_k}} \varrho\left(w_j,x\right) \le \frac{1}{2}\left[\varrho\left(w_j,X_1(\omega)\right) + \varrho\left(w_j,X_2(\omega)\right)\right]\right\}$$

and

$$A'_{i_1,i_2,\ldots,i_k} = A'_{i_1,i_2,\ldots,i_{k-1}} \cap \left(A_{i_1,i_2,\ldots,i_k} \setminus \bigcup_{l=1}^{i_k-1} A_{i_1,i_2,\ldots,i_{k-1},i_l}\right),$$

where $A'_{i_1} = A_{i_1} \setminus \bigcup_{l=1}^{i_1-1} A_l$. Then

$$A'_{i_1,i_2,\ldots,i_k} \in \mathcal{A}, \quad A'_{i_1,i_2,\ldots,i_k} \cap A'_{i'_1,i'_2,\ldots,i'_k} = \emptyset \text{ if } i_s \neq i'_s \text{ for some } 1 \le s \le k,$$

and, by condition (1)

$$\bigcup_{i_k=1}^{\infty} A'_{i_1,i_2,\ldots,i_k} = A'_{i_1,i_2,\ldots,i_{k-1}}; \quad \bigcup_{i_1=1}^{\infty} A'_{i_1} = \Omega.$$

Now choose $t_{i_1,i_2,\ldots,i_k} \in S_{i_1,i_2,\ldots,i_k}$ and define

$$T_k(\omega) = t_{i_1,i_2,\ldots,i_k} \text{ for } \omega \in A'_{i_1,i_2,\ldots,i_k}.$$

For all $\omega \in \Omega$ the sequence $\{T_n(\omega),\ n \geq 1\}$ satisfies the Cauchy condition and therefore converges to some $T(\omega) \in S$. By definition we have

$$\forall_{w_j \in W}\ \forall_{\omega \in \Omega} \quad \varrho\left(w_j, T_n(\omega)\right) \leq \frac{1}{2}\left[\varrho\left(w_j, X_1(\omega)\right) + \varrho\left(w_j, X_2(\omega)\right)\right] + \frac{1}{2^n}.$$

Using the fact that W is a dense set in S and taking the limit completes the proof. □

Proof of Theorem 1. It is easy to check that $d(\cdot,\cdot)$ is a metric. We will show that convergence in this metric is equivalent to the convergence in probability.

Note that for all $\varepsilon > 0$

$$\begin{aligned} d(X_n, X) &= E\varrho(X_n, X) \\ &= \int_{\varrho(X_n,X)>\varepsilon} \varrho(X_n, X)dP(\omega) + \int_{\varrho(X_n,X)\leq\varepsilon} \varrho(X_n, X)dP(\omega). \end{aligned}$$

So we always have

$$\begin{aligned} \varepsilon P(\varrho(X_n, X) > \varepsilon) &\leq d(X_n, X) \\ &\leq \varepsilon P(\varrho(X_n, X) \leq \varepsilon) + MP(\varrho(X_n, X) > \varepsilon). \end{aligned} \tag{7}$$

Now suppose that $d(X_n, X) \to 0$. Using the first inequality from (7) for all $\varepsilon > 0$ we obtain

$$P(\varrho(X_n, X) > \varepsilon) \leq \frac{d(X_n, X)}{\varepsilon} \to 0,$$

so $X_n \xrightarrow[n\to\infty]{P} X$.

Suppose that $X_n \xrightarrow[n\to\infty]{P} X$ and choose any $\varepsilon > 0$. Using the second inequality from (7) we have

$$d(X_n, X) \leq \varepsilon + MP(\varrho(X_n, X) > \varepsilon) \xrightarrow[n\to\infty]{} \varepsilon$$

because ε was chosen arbitrarily we have $d(X_n, X) \to 0$.

Now it is enough to show that $(\mathcal{X}, d)$ satisfies (2). Using Lemma 1 we have

$$\forall_{X_1, X_2 \in \mathcal{X}}\ \exists_{T \in \mathcal{X}}\ \forall_{z \in S}\ \forall_{\omega \in [0,1]} \quad \varrho\left(z, T(\omega)\right) \leq \frac{1}{2}\left[\varrho\left(z, X_1(\omega)\right) + \varrho\left(z, X_2(\omega)\right)\right].$$

For each $Z \in \mathcal{X}$ we can construct the sequence of simple r.e. convergent to Z. And taking expectation for both sides of inequality (respectively to ω) we obtain

$$\forall_{X_1, X_2 \in \mathcal{X}}\ \exists_{T \in \mathcal{X}}\ \forall_{Z \in \mathcal{X}} \quad d\left(Z, T\right) \leq \frac{1}{2}\left[d\left(Z, X_1\right) + \varrho\left(Z, X_2\right)\right]. \qquad \square$$

Note that if the space of values (S, ϱ) does not satisfy condition (6) we may introduce the equivalent metric $\varrho_1(x, y) = \frac{\varrho(x,y)}{1+\varrho(x,y)}$. The metric space (S, ϱ_1) satisfies condition (6) and we have the following:

Corollary 1. *Let (S,ϱ) be a separable, complete metric space, satisfying the following condition: for all $x_1, x_2 \in S$ there exists $t \in S$ such that for all $z \in S$*

$$[\varrho(x_1,z)-\varrho(t,z)]\,[1+\varrho(x_2,z)]+[\varrho(x_2,z)-\varrho(t,z)]\,[1+\varrho(x_1,z)]\geq 0.$$

Then $(\mathcal{X}_S,d)$, where $d(X,Y)=E\frac{\varrho(X,Y)}{1+\varrho(X,Y)}$ satisfies (2) and convergence in probability is equivalent to the convergence in metric.

Acknowledgements. The authors are most grateful to the referee for a careful reading and suggestions which have helped to improve the paper.

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Received April 27, 2007