ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXI, 2007 SECTIO A 23–38

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On the theorem of regularity of decrease for universal linearly invariant families of functions

ABSTRACT. This article includes results connected with the theorem of regularity of decrease for linearly invariant families $\mathcal{U}_\alpha$ of analytic functions in the unit disk. In particular the question about a cardinality of the set of directions of intensive decrease for any function from $\mathcal{U}_\alpha$ is considered.

In this article we study linearly invariant families, which were defined by Ch. Pommerenke [10] in 1964.

Definition 1 ([10])**.** A family $\mathfrak{M}$ of functions $f(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n$ analytic in the unit disk $\Delta = \{z : |z| < 1\}$ is called a linearly invariant family (LIF) if each function $f \in \mathfrak{M}$ satisfies the following conditions:

1) $f'(z) \neq 0$ for all $z \in \Delta$ (local univalence);
2) functions of the form

$$\Lambda[f(z)] = \frac{f\left(e^{i\theta}\frac{z+a}{1+\overline{a}z}\right) - f(e^{i\theta}a)}{f'(e^{i\theta}a)\cdot(1-|a|^2)e^{i\theta}} = z + \dots$$

for all $a \in \Delta$ and $\varphi \in \mathbb{R}$ belong to $\mathfrak{M}$ (invariance with respect to a conformal automorphism of the unit disk Δ).

2000 *Mathematics Subject Classification.* 30E35, 30C55, 30D99.

Key words and phrases. Locally univalence, linear invariance, linearly invariant family, theorem of regularity.

Definition 2 ([2])**.** The quantity

$$\operatorname{ord}\mathfrak{M} = \sup_{f\in\mathfrak{M}} |\, a_2(f)|$$

is called the order of the LIF $\mathfrak{M}$.

Definition 3 ([10])**.** The union of all linearly invariant families of order not greater than α is called a universal linearly invariant family of order α and it is denoted by $\mathcal{U}_\alpha$.

For every continuous function $g : \Delta \to \mathbb{C}$ and $r \in [0;1)$ we put

$$M(r,\phi) = \max_{|z|=r} |\, \phi(z)|, \quad m(r,\phi) = \min_{|z|=r} |\, \phi(z)|.$$

For linearly invariant families there is a list of theorems concerning the regularity growth. Statements of this type characterize the order of growth of moduli of functions and their derivatives.

Such results, for example, for the well-known class S of univalent functions in Δ were obtained in [1], [7], [8].

Theorem A (regularity of growth in S)**.** *Let $f \in S$. Then*

1) *there exists the limit*

$$\lim_{r\to 1-}\left[M(r,f)\frac{(1-r)^2}{r}\right] = \lim_{r\to 1-}\left[M(r,f')\frac{(1-r)^3}{1+r}\right] = \delta \in [0,1]$$

and $\delta = 1$ for the Koebe function $f_\theta(z) = z(1-ze^{-i\theta})^{-2}$ only. Functions under the sign of the limit are decreasing with respect to r, $0 < r < 1$, if $\delta \neq 1$.

2) *If $\delta \neq 0$, then there exists $\varphi^0 \in [0,2\pi)$ such that*

$$\lim_{r\to 1-}\left[|f(re^{i\varphi})|\frac{(1-r)^2}{r}\right] = \lim_{r\to 1-}\left[|f'(re^{i\varphi})|\frac{(1-r)^3}{1+r}\right] = \begin{cases}\delta, & \varphi = \varphi^0,\\ 0, & \varphi \neq \varphi^0;\end{cases}$$

functions under the sign of the limit are decreasing with respect to $r \in (0,1)$ too.

Theorems of regularity of growth for universal LIF were proved in [2], [11], [12] (see also the review [5]).

Theorem B (regularity of growth in $\mathcal{U}_\alpha$)**.** *Let $f \in \mathcal{U}_\alpha$. Then*

1) *for all $\varphi \in [0;2\pi)$ functions*

$$|f'(re^{i\varphi})|\frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}} \quad \text{and} \quad M(r,f')\frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}}$$

are non-increasing with respect to $r \in (0;1)$;

2) *there exist numbers $\delta^0 \in [0,1]$ and $\varphi^0 \in \mathbb{R}$ such that*

$$\delta^0 = \lim_{r\to 1-}\left[M(r,f)2\alpha\left(\frac{1-r}{1+r}\right)^\alpha\right] = \lim_{r\to 1-}\left[M(r,f')\frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}}\right]$$

$$= \lim_{r\to 1-}\left[|f'(re^{i\varphi^0})|\frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}}\right] = \lim_{r\to -1}\left[|f(e^{i\varphi^0})|2\alpha\left(\frac{1-r}{1+r}\right)^{\alpha}\right].$$

3) $\delta^0 = 1 \Longleftrightarrow f(z) = k_\theta(z) = \dfrac{e^{i\theta}}{2\alpha}\left[\left(\dfrac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right)^{\alpha} - 1\right]$, *$\theta \in \mathbb{R}$ is fixed.*

Taking into consideration the above-stated class of theorems it is natural to consider the question about the order of decrease of the analogous values.

In the paper [3] (see also [4]) the following theorem of regularity of decrease was proved.

Theorem 1 (regularity of decrease in $\mathcal{U}_\alpha$)**.** *Let $f \in \mathcal{U}_\alpha$. Then*

1) *there exist numbers $\delta_0 \in [1,\infty]$ and $\varphi_0 \in \mathbb{R}$ such that*

$$\delta_0 = \lim_{r\to 1-}\left[m(r,f')\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\right] = \lim_{r\to 1-}\left[|f'(re^{i\varphi_0})|\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\right].$$

Functions under the sign of the limit are non-decreasing with respect to $r \in (0;1)$ for all $\phi_0 \in \mathbb{R}$.

2) $\delta_0 = 1 \Longleftrightarrow f(z) = k_\theta(z) = -\dfrac{e^{i\theta}}{2\alpha}\left[\left(\dfrac{1-ze^{-i\theta}}{1+ze^{-i\theta}}\right)^{\alpha} - 1\right]$, *where $\theta \in \mathbb{R}$ is fixed.*

Definition 4. The number φ_0 from Theorem 1 we shall call a direction of maximal decrease (shortly written d.m.d.) of $f(z)$.

Definition 5. Every number $\theta \in [0;2\pi)$ such that

$$\lim_{r\to 1-}|f'(re^{i\theta})|\frac{(1+r)^{(\alpha+1)}}{(1-r)^{(\alpha-1)}} = \delta_\theta \in [1;\infty)$$

we shall call a direction of intensive decrease (shortly written d.i.d.) of $f(z)$.

It is natural to define the partition of family $\mathcal{U}_\alpha$ into disjoint classes $\mathcal{U}_\alpha(\delta_0)$, $\delta_0 \in [1;\infty]$, where the same number δ_0 (this is the number from Theorem 1) corresponds to all functions from the class $\mathcal{U}_\alpha(\delta_0)$.

Since the class $\mathcal{K} = \mathcal{U}_1$ has been well investigated, therefore, we will study the case $\alpha > 1$ only.

Theorem 2. *Let $\alpha > 1$, $f \in \mathcal{U}_\alpha(\delta_0)$, $\delta_0 < \infty$; θ is one of d.i.d. of the function f and $\delta \in [\delta_0,\infty)$ is a number which corresponds to this d.i.d.*

Denote by $\Delta(\eta)$ the Stoltz angle with measure 2η, $\eta \in (0,\frac{\pi}{2})$ with vertex at the point $e^{i\theta}$, $\Phi(\zeta) = \arg f'(\rho(\zeta)e^{i\theta})$, $\zeta \in \Delta(\eta)$ where

$$\rho(\zeta) = \sqrt{\frac{(1-r_0^2)^2}{4r_0^4c^2(\zeta)} + \frac{1}{r_0^2}} + \frac{1}{2c(\zeta)}\left(1 - \frac{1}{r_0^2}\right), \quad r_0 = \sin\eta,$$

$$c(\zeta) = \Re\{\zeta e^{-i\theta}\} - \tan\eta|\Im\{\zeta e^{-i\theta}\}|.$$

Then for all $n \in \mathbb{N}$ *if* $\alpha \notin \mathbb{N}$ *and for all* $n \in \mathbb{N}$ *such that* $n < \alpha + 1$ *if* $\alpha \in \mathbb{N}$*:*

$$\frac{f^{(n)}(\zeta)}{k_\theta^{(n)}(\zeta)} e^{-i\Phi(\zeta)} \xrightarrow[|\zeta|\to 1-]{} \delta$$

in $\Delta(\eta)$*.*

Thus, if a function $f \in \mathcal{U}_\alpha$ has d.i.d. θ, then for above-stated n a behavior of functions $|f^{(n)}|$ and $|k_\theta^{(n)}|\delta$ differs a little in the angle domain

$$\Delta(R, \eta) = \left\{ \zeta \in \Delta : \ |\arg(1 - \zeta e^{-i\theta})| < \eta, \ R < |\zeta| < 1 \right\}$$

as $R \to 1$.

Proof. For any $\phi \in [0; 2\pi)$ there exists the limit

$$\lim_{r \to 1-} \left[|f'(re^{i\phi})| \frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}} \right] = \delta(\phi).$$

Let us fix $a \in \Delta$ and ϕ and denote $z = \frac{re^{i\phi} - a}{1 - \overline{a} re^{i\phi}}$, $|z| = R(r)$. It is known [3, Th. 2] that $\lim_{r \to 1-} R'(r) = \frac{1 - |a|^2}{|1 - \overline{a} e^{i\phi}|}$. For such z we have

$$\begin{aligned}
&\lim_{r \to 1-} \left[|f'(z, a)| \frac{(1 + |z|)^{\alpha+1}}{(1 - |z|)^{\alpha-1}} \right] \\
&= \lim_{r \to 1-} \frac{|f'(re^{i\phi})|(1+r)^{\alpha+1}}{|f'(a)||1 + \overline{a} z|^2 (1-r)^{\alpha-1}} \lim_{r \to 1-} \left(\frac{1-r}{1 - R(r)} \right)^{\alpha-1} \\
&= \lim_{r \to 1-} \frac{\delta(\phi)}{|f'(a)| \left| 1 + \overline{a} \frac{e^{i\phi} - a}{1 - \overline{a} e^{i\phi}} \right|^2} \left(\lim_{r \to 1-} \frac{1}{R'(r)} \right)^{\alpha-1} \\
&= \frac{\delta(\phi) |1 - \overline{a} e^{i\phi}|^{2\alpha}}{|f'(a)|(1 - |a|^2)^{\alpha+1}} \\
&\geq \lim_{R(r) \to 1-} \left[m(R(r), f'(z, a)) \frac{(1 + R(r))^{\alpha+1}}{(1 - R(r))^{\alpha-1}} \right] = \delta_a.
\end{aligned}$$

Let us assume now ϕ to be equal θ — d.i.d. of $f(z)$. Put $a = \rho e^{i\theta}$, then

$$\frac{\delta(\theta)(1-\rho)^{2\alpha}}{|f'(\rho e^{i\theta})|(1-\rho^2)^{\alpha+1}} = \frac{\delta(\theta)(1-\rho)^{\alpha-1}}{|f'(\rho e^{i\theta})|(1+\rho)^{\alpha+1}} \geq \delta_a.$$

Therefore, $\delta_a \xrightarrow[\rho \to 1-]{} 1$ and in view of the Theorem 3 from [3] any locally convergent in Δ sequence $f_n(z) = f(z, \rho_n e^{i\theta})$ converges to $k_{\theta_1}(z)$ as $\rho_n \xrightarrow[n \to \infty]{} 1-$ for some $\theta_1 \in [0; 2\pi)$.

We shall prove $\theta_1 = \theta$. Denote $R_n = \frac{r + \rho_n}{1 + \rho_n r}$.

$$|k'_{\theta_1}(re^{i\theta})| = \lim_{n \to \infty} |f'_n(re^{i\theta})| = \lim_{n \to \infty} \frac{|f'(R_n e^{i\theta})|}{|f'(\rho_n e^{i\theta})|(1 + \rho_n r)^2}$$

$$= \lim_{n\to\infty} \frac{|f'(R_n e^{i\theta})| \frac{(1+R_n)^{\alpha+1}}{(1-R_n)^{\alpha-1}}}{|f'(\rho_n e^{i\theta})| \frac{(1+\rho_n)^{\alpha+1}(1+\rho_n r)^2}{(1-\rho_n)^{\alpha-1}}} \left(\lim_{n\to\infty} \frac{1-R_n}{1-\rho_n} \right)^{\alpha-1}$$

$$= \frac{\delta(\theta)}{\delta(\theta)(1+r)^2} \left(\frac{1-r}{1+r} \right)^{\alpha-1} \xrightarrow[r\to 1-]{} 0,$$

and this is possible in the case $\theta_1 = \theta$ only.

Taking into account the inequality (see [10])

$$\text{(1)} \qquad \frac{(1-r)^{\alpha-1}}{(1+r)^{\alpha+1}} \le |f'(z)| \le \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}, \quad r = |z|$$

one can use Vitali theorem for the functions $f'(z, \rho e^{i\theta})$. Thus

$$f'(z, \rho e^{i\theta}) \xrightarrow[\rho\to 1-]{} k'_\theta(z)$$

locally uniformly in Δ. In particular, for every fixed $r_0 \in (0,1)$

$$\frac{f'\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)}{f'(\rho e^{i\theta})(1+\rho e^{-i\theta}z)^2} \xrightarrow[\rho\to 1-]{} \frac{(1-ze^{-i\theta})^{\alpha-1}}{(1+ze^{-i\theta})^{\alpha+1}}$$

uniformly in the disk $\{|z| \le r_0\}$. Thus functions

$$\frac{f'\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)}{f'(\rho e^{i\theta})} \quad \text{and} \quad (1+\rho e^{-i\theta}z)^2 \frac{(1-ze^{-i\theta})^{\alpha-1}}{(1+ze^{-i\theta})^{\alpha+1}} = \frac{k'_\theta\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)}{k'_\theta(\rho e^{i\theta})}$$

converge uniformly in $\{|z| \le r_0\}$ to the same analytic function on $\{|z| \le r_0\}$ as $\rho \to 1-$, i.e.

$$\text{(2)} \qquad \frac{f'\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)}{f'(\rho e^{i\theta})} - \frac{k'_\theta\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)}{k'_\theta(\rho e^{i\theta})} \xrightarrow[\rho\to 1-]{} 0$$

uniformly in $\{|z| \le r_0\}$. Further proof of the case $n = 1$ follows the line proved for a similar theorem in [12].

The function $\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}$ maps univalently the disk $\{|z| \le r_0\}$ onto the disk with the center $c(r_0) = e^{i\theta}\rho\frac{1-r_0^2}{1-\rho^2 r_0^2}$ and the radius $r_*(r_0) = \frac{r_0(1-\rho^2)}{1-\rho^2 r_0^2}$. It follows that

$$\frac{f'(\zeta)}{k'_\theta(\zeta)} \frac{k'_\theta(\rho e^{i\theta})}{f'(\rho e^{i\theta})} \xrightarrow[\rho\to 1-]{} 1$$

uniformly in the disk $K_\rho(r_0) = \{|\zeta - c(r_0)| \le r_*(r_0)\}$. If we denote $\Phi(\rho) = \arg f'(\rho e^{i\theta})$, we have $\frac{f'(\zeta)}{k'_\theta(\zeta)} e^{-i\Phi(\rho)} \xrightarrow[\rho\to 1-]{} \delta$ uniformly in $K_\rho(r_0)$, thus for each $\varepsilon > 0$ there exists $R_1 \in (0,1)$ such that for all $\rho \in (R_1, 1)$

$$\text{(3)} \qquad \left| \frac{f'(\zeta)}{k'_\theta(\zeta)} e^{-i\Phi(\rho)} - \delta \right| < \varepsilon,$$

for all $\zeta \in K_\rho(r_0)$. Let 2β be the measure of an angle with the vertex at $e^{i\theta}$ and with the arms tangential to the circle $K_\rho(r_0)$. Then

$$\sin\beta = \frac{r_*(r_0)}{1-|c(r_0)|} = \frac{r_0(1-\rho^2)}{1-\rho^2 r_0^2 - \rho + \rho r_0^2} = \frac{r_0(1+\rho)}{1+\rho r_0^2} = \psi(\rho),$$

The function $\psi(\rho)$ is increasing. Consequently, for $\rho \in (R',1)$ the family of disks $K_\rho(r_0)$ covers a subset of Δ, which contains $\Delta(R,\eta)$ for some R and η. We can take $\arcsin r_0$, instead of η because $\psi(\rho)$ is an increasing function. Thus we can choose η arbitrarily close to $\pi/2$ for r_0 close to 1. Therefore, (3) holds in $\Delta(R,\eta)$, where $\eta \in (0,\pi/2)$ is fixed and R depends on ε. Then for every $\zeta \in \Delta(R,\eta)$ there is a $\Phi = \Phi(\rho)$ (not necessary one, because ζ belongs to many circles $K_\rho(r_0)$), where ρ is such that $\zeta \in K_\rho(r_0)$. Consequently, we can choose such disk $K_\rho(r_0)$ that ζ lies on a radius which is orthogonal to one of the sides of the sector $\Delta(R,\eta)$. Then $\sin\left(\frac{\pi}{2}-\eta\right) = \frac{\Im[(\zeta - c(r_0))e^{-i\theta}]}{|(\zeta-c(r_0))e^{-i\theta}|}$. Let us suppose that $\Im(\zeta e^{-i\theta}) \neq 0$, otherwise we can take ζ equal to center $c(r_0)$ of $K_\rho(r_\theta)$. Therefore

$$\frac{1}{\cos^2\eta} = \left(\frac{\Re(\zeta e^{-i\theta}) - |c(r_0)|}{\Im(\zeta e^{-i\theta})}\right)^2 + 1.$$

That is

$$\tan\eta = \frac{\Re(\zeta e^{-i\theta}) - |c(r_0)|}{|\Im(\zeta e^{-i\theta})|} \iff |c(r_0)| = \Re(\zeta e^{-i\theta}) - \tan\eta|\Im(\eta e^{-i\theta})|.$$

Since $|c(r_0)| = \rho\frac{1-r_0^2}{1-\rho^2 r_0^2}$, we get $\rho^2 r_0^2|c(r_0)| + \rho(1-r_0^2) - |c(r_0)| = 0$ and

$$\rho = \rho(\zeta) = \sqrt{\frac{(1-r_0^2)^2}{4r_0^4|c(r_0)|^2} + \frac{1}{r_0^2}} + \frac{1}{2|c(r_0)|}\left(1 - \frac{1}{r_0^2}\right),$$

where $r_0 = \sin\eta$. This proves the Theorem 2 in the case $n=1$.

Let now $n \geq 2$. After differentiating (2) with respect to z $n-1$ times, then multiplication by $(1+\rho e^{-i\theta}z)^2$, we get the expression

$$\frac{f^{(n)}\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)(1-\rho^2)^{n-1}}{f'(\rho e^{i\theta})} - \frac{k_\theta^{(n)}\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)(1-\rho^2)^{n-1}}{k'_\theta(\rho e^{i\theta})}$$

and passing to the limit as $\rho \to 1-$ we conclude that it tends to zero uniformly in $\{|z| \leq r_0\}$.

Since

$$\frac{k'_\theta\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)}{k'_\theta(\rho e^{i\theta})} = (1+\rho e^{-i\theta}z)^2\frac{(1-ze^{-i\theta})^{(\alpha-1)}}{(1+ze^{-i\theta})^{(\alpha+1)}},$$

then the function $\frac{k'_\theta\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)}{k'_\theta(\rho e^{i\theta})}$ is bounded away from zero as $\rho \to 1-$ in the disk $\{|z| \le r_0\}$. Since

$$\frac{k'_\theta\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)}{k'_\theta(\rho e^{i\theta})} \xrightarrow[\rho\to 1-]{} \left(\frac{1-ze^{-i\theta}}{1+ze^{-i\theta}}\right)^{\alpha-1}$$

locally uniformly in Δ, then

$$\frac{k_\theta^{(n)}\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)(1-\rho^2)^{n-1}}{k'_\theta(\rho e^{i\theta})}$$

$$\xrightarrow[\rho\to 1-]{} (-2)^{n-1}e^{-i(n-1)\theta}(\alpha-1)\dots(\alpha-(n-1))\left(\frac{1-ze^{-i\theta}}{1+ze^{-i\theta}}\right)^{\alpha-n}$$

locally uniformly in Δ, and thus locally uniformly in the disk $\{|z| \le r_0\}$. Consequently, the function $\frac{k_\theta^{(n)}\left(\frac{z+\rho e^{i\theta}}{1+\rho e^{-i\theta}z}\right)(1-\rho^2)^{n-1}}{k'_\theta(\rho e^{i\theta})}$ is bounded away from zero in the disk $\{|z| \le r_0\}$ firstly for any natural number n, if α is not natural, and, secondly, for every n such that $n < \alpha+1$, if α is natural.

Thus for above-stated n

$$\frac{f^{(n)}(\zeta)}{k_\theta^{(n)}(\zeta)}\frac{k'_\theta(\rho e^{i\theta})}{f'(\rho e^{i\theta})} \xrightarrow[\rho\to 1-]{} 1$$

uniformly in $K_\rho(r_0)$. Next, similarly as in the case $n=1$, from (4) we get $\frac{f^{(n)}(\zeta)}{k_\theta^{(n)}(\zeta)}e^{-i\Phi(\zeta)} \xrightarrow[\rho\to 1-]{} \delta$ uniformly in $\Delta(R,\eta)$ as $R \to 1-$.

Theorem 2 has been proved. □

We have now all necessary facts to give the answer to the following question: what is a cardinality of the set of d.i.d. for the function $f \in \mathcal{U}_\alpha$, $\alpha > 1$?

It was proved in [3] that if $f \in \mathcal{U}_\alpha$ then for all $\varphi \in [0; 2\pi)$ there exists $\delta(\varphi)$ such that for any circle (or straight line) Γ orthogonal to $\partial\Delta$ at the point $e^{i\varphi}$ there holds

$$|f'(\xi)|\frac{(1+|\xi|)^{\alpha+1}}{(1-|\xi|)^{\alpha-1}} \longrightarrow \delta(\varphi)$$

as $\xi \longrightarrow e^{i\varphi}$ along Γ and $\delta(\varphi)$ does not depend on Γ. This property is not true for $k_0(z)$, if Γ is not orthogonal to $\partial\Delta$. It follows from Theorem 2 that under assumptions concerning the curve Γ this property is false not only for the function k_0 but for arbitrary function $f \in \mathcal{U}_\alpha$ (which have any d.i.d.) either. Thus, if θ is the d.i.d of the function $f(z)$ then there exist two curves Γ_1 and Γ_2 in Δ such that $|f'(z')|\frac{(1+|z'|)^{\alpha+1}}{(1-|z'|)^{\alpha-1}} \to a'$ as $z' \to e^{i\theta}$ along Γ_1 and $|f'(z'')|\frac{(1+|z''|)^{\alpha+1}}{(1-|z''|)^{\alpha-1}} \to a''$ as $z'' \to e^{i\theta}$ along Γ_2. And $a' \ne a''$. Thus $e^{i\theta}$ is the point of indeterminacy of the function $|f'(z)|\frac{(1+|z|)^{\alpha+1}}{(1-|z|)^{\alpha-1}}$. By Bagemihl theorem (see [9]) the set of such points is at most countable. Therefore, the set of d.i.d. of a function $f\in\mathcal{U}_\alpha$ is at most countable.

It is natural to ask whether it is possible to give an example of a function, which has a given number of d.i.d. The following theorem gives the answer to this question.

Theorem 3. 1) *Let n be a fixed integer number, $n\geq 2$ and $1<\alpha<\infty$. Then a function*

$$g_{n,\alpha}(z)=\int\limits_0^z (1-s^n)^{\alpha-1}\,ds\in\mathcal{U}_\alpha$$

and possesses exactly n d.i.d.

2) *The function*

$$g_\alpha(z)=\int\limits_0^z\left[\frac{1-\exp\left(-\pi\frac{1-s}{1+s}\right)}{1-e^{-\pi}}\right]^{\alpha-1}\frac{1}{(1+s)^2}\,ds\in\mathcal{U}_\alpha$$

and the set of its d.i.d. is countable.

Remark 1. In the case $n=1$ $k_\theta\in\mathcal{U}_\alpha$ is a trivial example.

Remark 2. The similar result for directions of intensive growth is known (see [13]), it has been established however not for all $1<\alpha<\infty$.

Proof. 1) Denote by $e^{i\theta}$ one of the values of $\sqrt[n]{1}$. Then

$$\begin{aligned}\lim_{r\to1-}\left[|g'_{n,\alpha}(re^{i\theta})|\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\right]&=\lim_{r\to1-}\frac{(1-r^n)^{\alpha-1}(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\\&=n^{\alpha-1}2^{\alpha+1}\in(1;\infty).\end{aligned}$$

Thus, if ord $g_{n,\alpha}=\alpha$, then all $\theta\in[0,2\pi)$ will be d.i.d. of $g_{n,\alpha}$ and their quantity is equal to n exactly (because $e^{i\theta}$ is a value of $\sqrt[n]{1}$).

Thus our aim is to prove that ord $g_{n,\alpha}=\alpha$.

We prove firstly that

$$\text{ord}\; g_{n,\alpha}\leq\alpha. \tag{5}$$

Since the order of the family $\mathfrak{M}$ is given by

$$\text{ord}\;\mathfrak{M}=\sup_{f\in\mathfrak{M}}\sup_{z\in\Delta}\left|-\overline{z}+\frac{1-|z|^2}{2}\frac{f''(z)}{f'(z)}\right|$$

it follows that in order to (5) it is sufficient to show that the inequality

$$\left|-\overline{z}+\frac{1-|z|^2}{2}\frac{g''_\alpha(z)}{g'_\alpha(z)}\right|\leq\alpha$$

is valid on each circle $\{z : |z| = r\}$, $0 \le r < 1$, or equivalently

$$\left| -|z|^2 + \frac{1-|z|^2}{2} \frac{(1-\alpha)nz^n}{1-z^n} \right| \le \alpha|z|. \tag{6}$$

The function $w = \dfrac{z^n}{1-z^n}$ maps the circle $\{z : |z| = r\}$ onto a circle symmetric with respect to real axis. Hence, the function

$$-|z|^2 + \frac{1-|z|^2}{2} \frac{(1-\alpha)nz^n}{1-z^n}$$

also maps $\{z : |z| = r\}$ onto a circle symmetric with respect to real axis and intersects it in the points

$$A_n = -r^2 - \frac{(1-\alpha)n}{2} \frac{r^n}{1+r^n}(1-r^2)$$

and

$$B_n = -r^2 + \frac{(1-\alpha)n}{2} \frac{r^n}{1-r^n}(1-r^2).$$

Let us find M_r, the maximum of left-hand side of the inequality (6) on any circle $\{z : |z| = r\}$. Since $0 \le r < 1$ and $1 < \alpha < \infty$, then B_n is non-positive for all r. Let us consider all possibilities of location of points A_n and B_n:

a) If $A_n \ge -B_n$, then $M_r = A_n$. In this case we have

$$-r^2 + \frac{(\alpha-1)n}{2} \frac{r^n}{1+r^n}(1-r^2) \ge r^2 + \frac{(\alpha-1)n}{2} \frac{r^n}{1-r^n}(1-r^2). \tag{7}$$

But

$$\frac{(\alpha-1)n}{2} \frac{r^n}{1+r^n}(1-r^2) \le \frac{(\alpha-1)n}{2} \frac{r^n}{1-r^n}(1-r^2),$$

hence the condition (7) is not fulfilled. Thus the case a) does not hold.

b) If $A_n \le -B_n$, then $M_r = -B_n$. This is true always.

So, $M_r = -B_n = r^2 + \frac{(\alpha-1)n}{2}(1-r^2)\frac{r^n}{1-r^n}$. We will obtain first our inequality (6) in the case $n = 2$.

$$M_r = -B_2 = r^2 + (\alpha - 1)\frac{r^2}{1-r^2}(1-r^2) = \alpha r^2.$$

That is $\alpha r^2 \le \alpha r$. It is true for all $r \in [0; 1)$. Thus the inequality (6) has been proved for $n = 2$.

Now let $n > 2$. It is required to establish the inequality

$$M_r = -B_n = r^2 + \frac{(\alpha-1)n}{2} \frac{r^n}{1-r^n}(1-r^2) \le \alpha r. \tag{8}$$

To prove (8) it is sufficient to show that the sequence $\{-B_n\}$ decreases. We can write B_n in the form

$$-B_n = r^2 + \frac{(\alpha-1)}{2}(1-r^2)F(n),$$

where $F(x) = \frac{xr^x}{1-r^x}$. Since

$$F'(x) = \frac{(r^x + xr^x \ln r)(1-r^x) + xr^{2x}\ln r}{(1-r^x)^2} = \frac{r^x(1-r^x+\ln r^x)}{(1-r^x)^2}$$

then the sequence $\{-B_n\}$ decreases for any $r \in [0;1)$. Hence (5) is proved.

Here the strict inequality is impossible. For, if ord $g_{n,\alpha} = \alpha - \varepsilon < \alpha$, $\varepsilon > 0$, then by Theorem 1 there should be

$$\lim_{r\to 1-} \left[|g'(re^{i\theta})| \frac{(1+r)^{\alpha-\varepsilon+1}}{(1-r)^{\alpha-\varepsilon-1}} \right] \in [1,\infty],$$

the last limit actually is equal to

$$\lim_{r\to 1-} \frac{(1-r^n)^{\alpha-1}(1+r)^{\alpha+1-\varepsilon}}{(1-r)^{\alpha-1-\varepsilon}} = 0.$$

This is a contradiction. We have proved that ord $g_{n,\alpha} = \alpha$. The case 1) has been established.

2) We are going to prove that the function $g_\alpha(z)$ is the limit of $g'_{n,\alpha}(z,a_n)$ for odd n tending to infinity, where

$$a_{2n+1} = \frac{1-\sin\dfrac{\pi}{2n+1}}{\cos\dfrac{\pi}{2n+1}}.$$

Notice that $a_n \in \Delta$, because

$$a_{2n+1} = \frac{1-\sin\frac{\pi}{2n+1}}{\sqrt{1-\sin^2\frac{\pi}{2n+1}}} = \sqrt{\frac{1-\sin\frac{\pi}{2n+1}}{1+\sin\frac{\pi}{2n+1}}} < 1.$$

By Theorem 2 from [9] all d.i.d. of the function $g_{2n+1,\alpha}(z)$ will be transformed into any d.i.d. of the function $g_{2n+1,\alpha}(z,a_{2n+1})$ by the conformal automorphism $\frac{z+a_{2n+1}}{1+a_{2n+1}z}$ of the unit disk.

$$g'_{2n+1,\alpha}(z,a_{2n+1}) = \frac{g'_{2n+1,\alpha}\left(\frac{z+a_{2n+1}}{1+a_{2n+1}z}\right)}{g'_{2n+1,\alpha}(a_{2n+1})(1+a_{2n+1}z)^2}$$
$$= \left[\frac{1-\left(\frac{z+a_{2n+1}}{1+a_{2n+1}z}\right)^{2n+1}}{1-a_{2n+1}^{2n+1}} \right]^{\alpha-1} \cdot \frac{1}{(1+a_{2n+1}z)^2}.$$

We obtain now the function $g_\alpha(z)$. We calculate the limit

$$\lim_{n\to\infty} g'_{2n+1,\alpha}(z,a_{2n+1}) = \left(\frac{1-\exp\left(-\pi\frac{1-z}{1+z}\right)}{1-e^{-\pi}} \right)^{\alpha-1} \frac{1}{(1+z)^2} = g'_\alpha(z).$$

It gives

$$g_\alpha(z) = \int_0^z \left[\frac{1-\exp\left(-\pi\frac{1-s}{1+s}\right)}{1-e^{-\pi}}\right]^{\alpha-1} \frac{1}{(1+s)^2}\,ds.$$

Let us prove that $g_\alpha(z) \in \mathcal{U}_\alpha$. By the formula (1) the sequence of functions $g'_{2n+1,\alpha}(z, a_{2n+1})$ is uniformly bounded and it converges for all $z \in \Delta$. Then by Vitali theorem $g_\alpha(z)$ is a uniform limit. And $g_\alpha(z) \in \mathcal{U}_\alpha$ since $\mathcal{U}_\alpha$ is compact in the topology of uniform convergence.

We prove that the set of d.i.d. of the function $g_\alpha(z)$ is countable. The numerator in the brackets (in the expression of the function $g_\alpha(z)$) vanishes in the points $\frac{1+2ki}{1-2ki}$, $k \in \mathbb{Z}$. We shall prove that each $\theta_k = \arg\frac{1+2ki}{1-2ki}$, $k \in \mathbb{Z}$ is d.i.d. of the function $g_\alpha(z)$. For this purpose we shall calculate the limit

$$\lim_{r\to 1-}\left[|g'_\alpha(re^{i\theta_k})|\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\right] = \frac{4\pi^{\alpha-1}(1+4k^2)^{\alpha-1}}{|1+e^{i\theta_k}|^2}\cdot\left|\frac{e^\pi}{e^\pi-1}\right|^{\alpha-1} \in (1;\infty),$$

because of

$$\begin{aligned}\lim_{r\to 1-}\frac{\left|1-\exp\left(-\pi\frac{1-re^{i\theta_k}}{1+re^{i\theta_k}}\right)\right|}{1-r} &= \lim_{r\to 1-}\frac{\left|1-\exp\left(-\pi\frac{1-2ki-r(1+2ki)}{1-2ki+r(1+2ki)}+2k\pi i\right)\right|}{1-r}\\ &= \lim_{r\to 1-}\frac{\left|\pi\frac{(1+4k^2)(1-r)}{1-2ki+r(1+2ki)}+o(1-r)\right|}{1-r}\\ &= \frac{\pi(1+4k^2)}{2}.\end{aligned}$$

Thus all θ_k are d.i.d. of $g_\alpha(z)$. The theorem has been proved. □

Our further purpose is to find a relationship between $\mathcal{U}_\alpha(\delta)$ for various δ. Next two theorems assert that it is possible to construct a function $f(z) \in \mathcal{U}_\alpha(\delta_1)$ (for given δ_1) using the given function $f(z) \in \mathcal{U}_\alpha(\delta_2)$ if certain conditions are satisfied.

Theorem 4. *If $f \in \mathcal{U}_\alpha(\delta_0)$ and $\delta_0 \in (1,\infty)$, then for all $\delta \in (1,\delta_0]$ there exists $a \in \Delta$ such that the function*

$$f(z,a) = \frac{f\left(\frac{z+a}{1+\bar{a}z}\right)-f(a)}{f'(a)(1-|a|^2)}$$

belongs to $\mathcal{U}_\alpha(\delta)$.

Proof. For all $\varphi \in [0;2\pi)$ there exists the limit

$$\lim_{r\to 1-}\left[|f'(re^{i\varphi})|\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\right] = \delta(\varphi).$$

Let us fix φ. Denote $z = \frac{re^{i\varphi}-a}{1-\overline{a}re^{i\varphi}}$, $|z| = R(r)$. For such z we will consider the limit

$$\begin{aligned}
\lim_{r\to1-}\left[|f'(z,a)|\frac{(1+R(r))^{\alpha+1}}{(1-R(r))^{\alpha-1}}\right] &= \lim_{r\to1-}\left[\frac{\left|f'\left(\frac{z+a}{1+\overline{a}z}\right)\right|}{|f'(a)||1+\overline{a}z|^2}\frac{(1+R(r))^{\alpha+1}}{(1-R(r))^{\alpha-1}}\right]\\
&= \delta(\varphi)\lim_{r\to1-}\left[\frac{|f'(re^{i\varphi})|}{|f'(a)||1+\overline{a}z|^2}\right]\cdot\left[\lim_{r\to1-}\frac{1}{R'(r)}\right]^{\alpha-1}\\
&= \frac{\delta(\varphi)}{|f'(a)|}\frac{|1-\overline{a}e^{i\varphi}|^{2\alpha}}{(1-|a|^2)^{\alpha+1}}\\
&\geq \lim_{R(r)\to1-}\left[m(R(r),f'(z,a))\frac{(1+R(r))^{\alpha+1}}{(1-R(r))^{\alpha-1}}\right] = \delta_a.
\end{aligned}$$

Put φ equal to φ_0 — d.m.d. of the function $f(z)$ and $a = \rho e^{i\varphi_0}$. Then $\delta(\varphi) = \delta_0$ and

$$\frac{\delta_0(1-\rho)^{2\alpha}}{|f'(a)|(1-\rho^2)^{\alpha+1}} = \frac{\delta_0(1-\rho)^{\alpha-1}}{|f'(\rho e^{i\varphi_0})|(1+\rho)^{\alpha+1}} \geq \delta_a. \tag{9}$$

For the fixed $a = \rho e^{i\varphi_0}$ there exists $\varphi_1 \in [0;2\pi)$ — d.m.d. of the function $f(z,a)$ such that

$$\begin{aligned}
\delta_a &= \lim_{r\to1-}\left[|f'(re^{i\varphi_1},a)|\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\right]\\
&= \lim_{r\to1-}\left[\frac{f'\left(\frac{re^{i\varphi_1}+a}{1+\overline{a}re^{i\varphi_1}}\right)}{|f'(a)||1+\overline{a}re^{i\varphi_1}|^2}\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\right].
\end{aligned}$$

If we denote $R_1(r)e^{i\gamma_1(r)} = \frac{re^{i\varphi_1}+a}{1+\overline{a}re^{i\varphi_1}}$, where $\gamma_1(r)$ is a real function, then we obtain

$$\begin{aligned}
\delta_a &\geq \lim_{r\to1-}\left[\frac{m(R_1(r),f'(z))}{|f'(a)||1+\overline{a}re^{i\varphi_1}|^2}\frac{(1-r)^{\alpha+1}}{(1+r)^{\alpha-1}}\right]\\
&= \frac{\delta_0}{|f'(a)||1+\overline{a}e^{i\varphi_1}|^{2\alpha}}\left(\frac{1-|a|^2}{|1+\overline{a}e^{i\varphi_1}|^2}\right)^{\alpha-1}\\
&= \frac{\delta_0(1-|a|^2)^{\alpha-1}}{|f'(a)||1+\overline{a}e^{i\varphi_1}|^{2\alpha}}\\
&\geq \frac{\delta_0(1-\rho^2)^{\alpha-1}}{|f'(a)|(1+\rho)^{2\alpha}} = \frac{\delta_0(1-\rho)^{\alpha-1}}{|f'(a)|(1+\rho)^{\alpha+1}}.
\end{aligned}$$

Therefore, putting $a = \rho e^{i\varphi_0}$, from (9) we get

$$\delta_a = \frac{\delta_0(1-\rho)^{\alpha-1}}{|f'(\rho e^{i\varphi_0})|(1+\rho)^{\alpha+1}}.$$

Since $|f'(\rho e^{i\varphi_0})|\frac{(1+\rho)^{\alpha+1}}{(1-\rho)^{\alpha-1}}$ is a non-decreasing function of $\rho \in (0;1)$, there exists ρ at which it takes the value $\delta_a \in (1;\delta_0]$. The theorem has been proved. □

Theorem 5. *If $f \in \mathcal{U}_\alpha(\delta_0)$, $\delta_0 \in (1,\infty)$, $\alpha > 1$ and there exists an interval $(x', x'') \subset [0;2\pi)$ which does not contain d.m.d. of the function $f(z)$, then for any $\delta \in (1;\infty)$ there exists a number $a \in \Delta$ such that $f(z,a) \in \mathcal{U}_\alpha(\delta)$.*

Proof. Let $\eta > 0$ be such that $x' + \eta = x_1 < x_2 = x'' - \eta$. By Privalov theorem of uniqueness (see [6]) there does not exist such $K > 0$ that $|f'(z)| \cdot (1-|z|)^{\alpha+1} > K$ in the sector $\{z : z \in \Delta,\ x_1 < \arg z < x_2\}$. Therefore there exists a sequence $a_n = \rho_n e^{i\theta_n}$, $\theta_n \in (x_1,x_2)$, $\theta_n \to \theta_0 \in [x_1,x_2]$, $\rho_n \xrightarrow[n\to\infty]{} 1$ such that $|f'(a_n)| = \frac{K_n}{(1-\rho_n)^{\alpha+1}}$, where $K_n \xrightarrow[n\to\infty]{} 0$.

Let us denote by φ_n — d.i.d. $f(z,a_n)$;

$$\frac{re^{i\varphi_n} + a_n}{1 + \overline{a}re^{i\varphi_n}} = R_n(r) \cdot e^{i\gamma_n(r)},$$

$\gamma_n(r)$ is a real function;

$$e^{i\gamma_n} \stackrel{\text{def}}{=} \frac{e^{i\varphi_n} + a_n}{1 + \overline{a_n}e^{i\varphi_n}}; \quad \delta_n^* \stackrel{\text{def}}{=} \lim_{r\to 1-}\left[|f'(re^{i\varphi_n}, a_n)|\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\right];$$

$$\delta_n \stackrel{\text{def}}{=} \lim_{r\to 1-}\left[|f'(re^{i\gamma_n})|\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\right].$$

We will find a connection between δ_n^* and δ_n:

$$\begin{aligned}
\delta_n^* &= \left[\lim_{r\to 1-}\frac{|f'(R_n(r)e^{i\gamma_n(r)})|}{|f'(a_n)||1+\overline{a_n}re^{i\varphi_n}|^2}\frac{(1+R_n(r))^{\alpha+1}}{(1-R_n(r))^{\alpha-1}}\right] \cdot \left(\lim_{r\to 1-}\frac{1-R_n(r)}{1-r}\right)^{\alpha-1} \\
&= \lim_{r\to 1-}\left[\frac{|f'(re^{i\gamma_n})|}{|f'(a_n)||1+\overline{a_n}re^{i\varphi_n}|^2}\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}\right] \cdot \left(\lim_{r\to 1-} R_n'(r)\right)^{\alpha-1} \\
&= \frac{\delta_n}{|f'(a_n)||1+\overline{a_n}e^{i\varphi_n}|^2} \cdot \left(\frac{1-|a_n|^2}{|1+\overline{a_n}e^{i\varphi_n}|^2}\right)^{\alpha-1} \\
&= \frac{\delta_n(1-\rho_n^2)^{\alpha-1}}{|f'(a_n)||1+\overline{a_n}e^{i\varphi_n}|^{2\alpha}} \\
&= \frac{\delta_n|1-\rho_n e^{i(\gamma_n-\theta_n)}|^{2\alpha}}{|f'(a_n)|(1-\rho_n^2)^{\alpha+1}} = \frac{\delta_n|1-\rho_n e^{i(\gamma_n-\theta_n)}|^{2\alpha}}{K_n(1+\rho_n)^{\alpha+1}} < \infty,
\end{aligned}$$

because of

$$1 + \overline{a_n}e^{i\varphi_n} = 1 + \overline{a_n} \cdot \frac{e^{i\gamma_n} - a_n}{1 - \overline{a_n}e^{i\gamma_n}} = \frac{1-\rho_n^2}{1-\rho_n e^{i(\gamma_n-\theta_n)}}\ .$$

From the sequence $\{a_n\}$ it is possible to choose the subsequence such that corresponding subsequences $\{\delta_n\}$ and $\{\delta_n^*\}$ will be convergent. Let us denote them as the initial sequences. Then

$$\lim_{n\to\infty}\delta_n^* = \lim_{n\to\infty}\delta_n \cdot \lim_{n\to\infty}\frac{|1-\rho_n e^{i(\gamma_n-\theta_n)}|^{2\alpha}}{K_n(1+\rho_n)^{\alpha+1}} \geq \lim_{n\to\infty}\delta_n \cdot \lim_{n\to\infty}\frac{|1-\rho_n e^{i\eta}|^{2\alpha}}{K_n(1+\rho_n)^{\alpha+1}},$$

because γ_n is d.i.d. of the function $f(z)$ by Theorem 2 in [3] (see also [4]) and, therefore, $\gamma_n \notin (x', x'')$. Since $K_n \xrightarrow[n\to\infty]{} 0$, then $\delta_n^* \xrightarrow[n\to\infty]{} \infty$.

Thus for any number $\delta \in (1;\infty)$ we can find n such that $\delta_n^* > \delta$. Then (by Theorem 4) for the function $f_n = f(z, a_n) \in \mathcal{U}_\alpha(\delta_n^*)$ there exists a number $a \in \Delta$ such that $f_n(z,a) \in \mathcal{U}_\alpha(\delta)$. The theorem has been proved. □

To establish the relationship between classes $\mathcal{U}_\alpha(\delta)$ we will need the following theorem.

Theorem 6. *For any function $f \in \mathcal{U}_\alpha(\delta_0)$, $\delta_0 \in [1;\infty]$ and for any function $\delta^*(\lambda)$, $\lambda \in (0;1)$ with values in $[\delta_0;\infty]$ there exists a family of functions $\psi_\lambda \in \mathcal{U}_\alpha(\delta^*(\lambda))$ such that $\psi_\lambda(z) \to f(z)$ locally uniformly in Δ as $\lambda \to 0$.*

Proof. It was shown in [10] that if $f_\lambda(z) \in \mathcal{U}_\alpha$, $f \in \mathcal{U}_\alpha$ and $\psi'_\lambda(z) = (f'(z))^{1-\lambda}(f'_\lambda(z))^\lambda$, then for any $\lambda \in (0;1)$ functions $\psi_\lambda(z) \in \mathcal{U}_\alpha$.

For all $\lambda \in (0;1)$ we select a function f_λ, satisfying the following conditions:

1) d.m.d. of the function $f_\lambda(z)$ is equal to d.m.d. of the function $f(z)$. We can achieve it by rotation $e^{-i\theta}f(ze^{i\theta})$;

2) $f_\lambda \in \mathcal{U}_\alpha(\delta(\lambda))$, where

$$\delta(\lambda) = \delta_0\left(\frac{\delta^*(\lambda)}{\delta_0}\right)^{\frac{1}{\lambda}} \in [\delta_0;\infty]$$

for $\lambda \in (0;1)$. Such a function exists, because $\mathcal{U}_\alpha(\delta(\lambda)) \neq \emptyset$. It follows from Theorem 5 and example of the function k_θ. In the case of $\delta(\lambda) = \infty$ we can take the function $f_\lambda(z) = z$.

With such choices of functions $f_\lambda(z)$, $\lambda \in (0;1)$ we have that

$$\psi_\lambda \in \mathcal{U}(\delta_0^{1-\lambda}\cdot\delta^\lambda(\lambda)) = \mathcal{U}(\delta^*(\lambda)).$$

We prove that $\psi_\lambda(z) \xrightarrow[\lambda\to 0]{} f(z)$ locally uniformly in Δ. Indeed, taking into account (1) we get that $f'(z)$ and $f'_\lambda(z)$ are bounded away from zero in Δ. Therefore

$$\left(\frac{f'_\lambda(z)}{f'(z)}\right)^\lambda \xrightarrow[\lambda\to 0]{} 1$$

locally uniformly in Δ. Hence

$$\psi'_\lambda = f'\cdot\left(\frac{f'_\lambda}{f'}\right)^\lambda \xrightarrow[\lambda\to 0]{} f'$$

locally uniformly in Δ. It means that for any $\varepsilon > 0$ there exists a number $N \in (0;1)$ such that as $\lambda < N$, $|\psi'_\lambda(z) - f'(z)| < \varepsilon$ for any $z \in K$, where K is a compact subset of Δ. Then for any $\varepsilon_1 > 0$ as $\lambda < N$ for any $z \in K$

$$|\psi_\lambda(z) - f(z)| = \left| \int_0^z (f'(s) - \psi'_\lambda(s))\, ds \right| \le \varepsilon \cdot C_K = \varepsilon_1,$$

because of $|z| < C_K$ for $z \in K$. Therefore, $\psi_\lambda(z) \xrightarrow[\lambda \to 0]{} f(z)$ uniformly in $K \subset \Delta$, that is locally uniformly in Δ. The theorem has been proved in all cases. □

If we put $\delta^*(\lambda) \equiv \delta \in [\delta_0, \infty]$ in Theorem 6, we get

Corollary. *For any function $f \in \mathcal{U}_\alpha(\delta_0)$, $\delta_0 \in [1, \infty]$ and $\delta \in [\delta_0, \infty]$ there is a family of functions $\psi_\lambda \in \mathcal{U}_\alpha(\delta)$, $\lambda \in (0;1)$ such that $\psi_\lambda(z) \to f(z)$ locally uniformly in Δ as $\lambda \to 0$.*

Let us notice that the requirement of $\delta \in [\delta_0, \infty]$ is essential. Namely for $\delta \in (1; \delta_0)$ and any function $f(z) \in \mathcal{U}_\alpha(\delta_0)$ there is no sequence of functions $f_n \in \mathcal{U}_\alpha(\delta)$ such that $f_n \xrightarrow[n \to \infty]{} f(z)$ locally uniformly in Δ.

Indeed, assume that there is this such a sequence f_n. The function $m(r, f')\frac{(1+r)^{\alpha+1}}{(1-r)^{\alpha-1}}$ is non-decreasing with respect to $r \in (0;1)$. Hence there is $r_0 \in (0;1)$ such that

$$m(r_0, f')\frac{(1+r_0)^{\alpha+1}}{(1-r_0)^{\alpha-1}} > \delta_0 - \frac{\delta_0 - \delta}{3}.$$

It follows from the uniform convergence of $f_n(z)$ that it is possible to choose $\varepsilon > 0$ and a natural n such that

$$|m(r_0, f'_n) - m(r_0, f')| < \varepsilon\,, \quad \varepsilon\frac{(1+r_0)^{\alpha+1}}{(1-r_0)^{\alpha-1}} < \frac{\delta_0 - \delta}{3}.$$

Then

$$|m(r_0, f'_n) - m(r_0, f')|\frac{(1+r_0)^{\alpha+1}}{(1-r_0)^{\alpha-1}} < \varepsilon\frac{(1+r_0)^{\alpha+1}}{(1-r_0)^{\alpha-1}} < \frac{\delta_0 - \delta}{3},$$

therefore,

$$\begin{aligned} m(r_0, f'_n)\frac{(1+r_0)^{\alpha+1}}{(1-r_0)^{\alpha-1}} &> m(r_0, f')\frac{(1+r_0)^{\alpha+1}}{(1-r_0)^{\alpha-1}} + \frac{\delta - \delta_0}{3} \\ &> \delta_0 + \frac{2}{3}(\delta - \delta_0) = \frac{2}{3}\delta + \frac{1}{3}\delta_0 \\ &> \frac{2}{3}\delta + \frac{1}{3}\delta = \delta, \end{aligned}$$

which contradicts to $f_n \in \mathcal{U}_\alpha(\delta)$.

Therefore, it follows that classes $\mathcal{U}_\alpha(\delta)$ extend with increase of δ, as if $\delta_1 \leq \delta_2$ then we can approximate functions from class $\mathcal{U}_\alpha(\delta_1)$ by functions from $\mathcal{U}_\alpha(\delta_2)$ and it is impossible to do so in the opposite direction.

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Received April 19, 2007