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Univalent anti-analytic perturbations of convex analytic mappings in the unit disc

Abstract. Let S_H be the class of normalized univalent harmonic mappings in the unit disc. We introduce subclasses of S_H, by choosing only these functions whose analytic parts are convex functions. For such mappings we establish coefficient, growth and distortion estimates. We also give solutions to covering problems. Obtained results are different from those, which are known or conjectured in the full class S_H.

1. Introduction. A function f is said to be a complex-valued harmonic function in a simply connected domain Ω in the complex plain $\mathbb{C}$ if both $\mathrm{Re}\{f\}$ and $\mathrm{Im}\{f\}$ are real harmonic in Ω. Every such f can be uniquely represented as

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in Ω with $g(0) = 0$.

A complex-valued harmonic function f, not identically constant, satisfying (1.1) is said to be sense-preserving in Ω if, and only if it satisfies the equation

$$g' = \omega h',$$

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where ω is analytic in Ω with $|\omega(z)| < 1$, $z \in \Omega$. The function ω is called the second complex dilatation of f. It is closely related to the Jacobian of f defined as follows

$$J_f(z) := |h'(z)|^2 - |g'(z)|^2, \quad z \in \Omega.$$

Recall that the necessary and sufficient condition for f to be locally univalent and sense-preserving in Ω is $J_f(z) > 0$, $z \in \Omega$. This is an immediate consequence of Levy's theorem (see [7]). Observe, that if $J_f(z) > 0$, then $|h'(z)| > 0$ and hence $g'(z)/h'(z)$ is well defined for every $z \in \Omega$. Thus the dilatation ω of locally univalent and sense-preserving function f in Ω can be expressed as

$$\omega(z) = \frac{g'(z)}{h'(z)}, \quad z \in \Omega. \tag{1.2}$$

Let $\Delta(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$, where $a \in \mathbb{C}$ and $r > 0$. Choose $\Omega = \Delta$, where $\Delta := \Delta(0, 1)$ is the unit disc in $\mathbb{C}$. Then every f satisfying (1.1) in Δ is uniquely determined by coefficients of the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \Delta, \tag{1.3}$$

where $a_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$ and $b_n \in \mathbb{C}$, $n = 1, 2, 3, \dots$. More information about harmonic mappings in the plane can be found in e.g. [3].

Clunie and Sheil-Small introduced in [1] the family S_H of all univalent and sense-preserving harmonic functions f satisfying (1.1) in Δ, such that $h(0) = 0$ and $h'(0) = 1$. In [6] we were studying properties of a subset of S_H consisting of all univalent anti-analytic perturbations of the identity in the unit disc. The main idea of this paper is to consider more general classes than the one introduced in [6]. Let $\alpha \in [0, 1)$. We define the class $\widehat{S}^\alpha$ of all $f \in S_H$, such that $|b_1| = \alpha$ and $h \in C$, where C denotes the well-known family of normalized univalent analytic functions which are convex. Additionally, we denote

$$\widehat{S} := \bigcup_{\alpha \in [0,1)} \widehat{S}^\alpha.$$

Note, that the dilatation ω of $f \in S_H$ is an analytic function satisfying (1.2) in Δ. Since $b_1 = \omega(0)$ and $|\omega(z)| < 1$, $z \in \Delta$, then we have the estimate $|b_1| < 1$. This explains why we have taken $\alpha \in [0, 1)$. Throughout this paper α will always mean a fixed number from $[0, 1)$.

The main results of this paper are solutions to some extremal problems in $\widehat{S}^\alpha$ and $\widehat{S}$. We establish coefficient, distortion and growth estimates. In particular, we derive the solution to covering problem.

2. Preliminary notes and examples. Let B be the set of all functions ϕ analytic in Δ such that $\phi(\Delta) \subset \overline{\Delta}$, where $\overline{\Delta} := \{z \in \mathbb{C} : |z| \le 1\}$. As it was mentioned earlier the dilatation ω of $f \in S_H$ belongs to B. Hence, some results concerning B will be useful in the study of $\widehat{S}^\alpha$ and $\widehat{S}$.

Let $\phi \in B$ and $\phi(0) = 0$. It is well known that, by the use of the maximum modulus principle, we can obtain

$$|\phi(z)| \le |z|, \quad z \in \Delta. \tag{2.1}$$

From (2.1) we can easily deduce that

$$|\phi'(0)| \le 1. \tag{2.2}$$

The inequalities (2.2) and (2.1) together are called the Schwarz lemma. In both of them the equality holds only for the function $\Delta \ni z \mapsto e^{i\theta} z$, where $\theta \in \mathbb{R}$ is constant (see [4]).

We will also need the following result due to Schur.

Theorem 2.1 ([5])**.** *If $\phi \in B$ and*

$$S_k(z) := \sum_{j=0}^{k} \lambda_j z^j, \quad \phi(z) = \sum_{n=0}^{\infty} \lambda_n z^n, \quad z \in \Delta \tag{2.3}$$

for $k = 0, 1, 2, \dots,$ then

$$\sum_{k=0}^{n} |S_k(z)|^2 \le n + 1, \quad z \in \Delta. \tag{2.4}$$

Next theorem due to Clunie and Sheil-Small gives us very important description of the class $\widehat{S}$.

Theorem 2.2 ([1])**.** *If f is a harmonic locally univalent and sense-preserving function in Δ satisfying* (1.1) *and for some ϵ ($|\epsilon| \le 1$), $h + \epsilon g$ is convex, then the function f is univalent in Δ and $f(\Delta)$ is a close-to-convex set.*

Because every univalent function is locally univalent, we have the following immediate corollary from Theorem 2.2.

Corollary 2.3. *Assume f is a harmonic function satisfying* (1.1), (1.2) *in Δ and $h \in C$. Then f is harmonic close-to-convex function if one of the following equivalent conditions hold:*

a) *$f \in \widehat{S}$;*
b) *$|\omega(z)| < 1$, $z \in \Delta$;*
c) *$J_f(z) > 0$, $z \in \Delta$.*

Let ω be the dilatation of $f \in \widehat{S}^\alpha$ given by (1.1). By the definition of ω we have $g' = \omega h'$ and hence f has the following integral representation

$$f(z) = h(z) + \overline{\int_0^z \omega(\zeta) h'(\zeta)\, \mathrm{d}\zeta}, \quad z \in \Delta, \tag{2.5}$$

where $h \in C$ and $\omega \in B_\alpha := \{\phi \in B : |\phi(0)| = \alpha\}$.

Moreover, since $h \in C$, then $\mathrm{Re}\{h(z)/z\} > 1/2$, $z \in \Delta$ (see [8]). According to the Riesz–Herglotz representation (see [5]) for the function $\Delta \ni z \mapsto 2h(z)/z - 1$, there exists a nondecreasing function μ in $[0, 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$, such that

$$h(z) = \int_0^{2\pi} \frac{z\, \mathrm{d}\mu(t)}{1 - e^{-it}z}, \quad z \in \Delta \tag{2.6}$$

and

$$h'(z) = \int_0^{2\pi} \frac{\mathrm{d}\mu(t)}{(1 - e^{-it}z)^2}, \quad z \in \Delta. \tag{2.7}$$

Putting (2.6) and (2.7) into (2.5) we obtain

$$f(z) = \int_0^{2\pi} \left[\frac{z}{1 - e^{-it}z} + \overline{\int_0^z \frac{\omega(\zeta)}{(1 - e^{-it}\zeta)^2}\, \mathrm{d}\zeta} \right] \mathrm{d}\mu(t), \quad z \in \Delta. \tag{2.8}$$

An important question is whether the families $\widehat{S}^\alpha$ and $\widehat{S}$, introduced in this paper, are normal and compact or not. Before we answer, we first give the following.

Example 2.4. For every $n = 0, 1, 2, \ldots$ the function

$$f_n(z) := z + \frac{n}{n+1}\overline{z}$$

is an univalent affine mapping in Δ. Thus f_n belongs to $\widehat{S}^\alpha$ with $\alpha = n/(n+1)$. The sequence $\{f_n\}$ converges locally uniformly in Δ to the function $f(z) := 2\,\mathrm{Re}\{z\}$, which is not univalent, hence neither $f \in \widehat{S}$ nor $f \in S_H$.

Theorem 2.5. *The family $\widehat{S}^\alpha$ is normal and compact. The family $\widehat{S}$ is normal but not compact.*

Proof. Both $\widehat{S}^\alpha$ and $\widehat{S}$ are normal as subclasses of the normal family S_H. The class $\widehat{S}$ is not compact, as it is shown in Example 2.4. The compactness of $\widehat{S}^\alpha$ follows from the representation (2.5) and the compactness of the classes C and B_α. □

Now we construct an example of a function, which seems to be extremal in many problems concerning $\widehat{S}^\alpha$.

Example 2.6. Let $\zeta \in \Delta$. Consider a function $f_\zeta = h_\zeta + \overline{g_\zeta}$ such that

$$h_\zeta(z) = \frac{z}{1-z}$$

and suppose that its dilatation ω_ζ satisfies

$$\omega_\zeta(z) = \frac{z+\zeta}{1+\zeta z}.$$

Now, from the identity (1.2) we have

$$g'_\zeta(z) = \frac{z+\zeta}{(1+\zeta z)(1-z)^2}$$

and since $g_\zeta(0) = 0$, by integration, we uniquely determine

$$g_\zeta(z) = \frac{z}{1-z} + \frac{\zeta - 1}{\zeta + 1}\left(\operatorname{Log}\frac{1+\zeta z}{1-z}\right).$$

Obviously, $|\omega_\zeta(z)| < 1$, $z \in \Delta$ so, in view of Corollary 2.3, the construction method assures that $f_\zeta \in \widehat{S}^\alpha$ for $\alpha = |\zeta|$.

3. Main results. Let $f \in \widehat{S}^\alpha$. By definition, the analytic part h of f belongs to C. Then from the theory of univalent analytic functions we have the following coefficient estimate

$$|a_n| \le 1, \quad n = 2, 3, 4, \dots. \tag{3.1}$$

Our first aim is to give an estimate on the coefficients b_n of g, where $\overline{g}$ is the anti-analytic part of f.

Theorem 3.1. *If $f \in \widehat{S}^\alpha$ and f is given by* (1.1), (1.3), *then*

$$|b_n| \le \frac{\alpha + \sqrt{(n-\alpha^2)(n-1)}}{n} \tag{3.2}$$

for $n = 2, 3, 4, \dots$.

Proof. Let ω be the dilatation of $f = h + \overline{g}$, where h, g are given by (1.3) and let

$$\omega(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \Delta, \tag{3.3}$$

where $c_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$ and $|c_0| = \alpha$. From the formula (2.8) we derive

$$a_n = \int_0^{2\pi} e^{-i(n-1)t}\, \mathrm{d}\mu(t), \quad n = 1, 2, 3, \dots \tag{3.4}$$

and

$$nb_n = \int_0^{2\pi} \left(\sum_{k=1}^{n} k e^{-i(k-1)t} c_{n-k}\right) \mathrm{d}\mu(t), \quad n = 1, 2, 3, \dots. \tag{3.5}$$

The formula (3.4) leads to the known estimate (3.1), whereas (3.5) yields

$$|nb_n| \le \max\left\{\left|\sum_{k=1}^{n} ke^{-i(k-1)t}c_{n-k}\right| : t \in [0, 2\pi]\right\}. \tag{3.6}$$

Observe, that

$$\begin{aligned}\sum_{k=1}^{n} ke^{-i(k-1)t}c_{n-k} &= e^{-i(n-1)t}\sum_{k=1}^{n} ke^{i(n-k)t}c_{n-k} \\ &= e^{-i(n-1)t}\sum_{k=0}^{n-1} S_k(e^{it}),\end{aligned} \tag{3.7}$$

where

$$[0, 2\pi] \ni t \mapsto S_k(e^{it}) := \sum_{j=0}^{k} c_j e^{ijt}$$

for $k = 0, 1, 2, \dots$. By applying (3.7) to (3.6) we obtain

$$|nb_n| \le \left|\sum_{k=0}^{n-1} S_k(e^{it})\right| \le \sum_{k=0}^{n-1} \left|S_k(e^{it})\right|.$$

Since $|S_0(e^{it})| = \alpha$, $t \in [0, 2\pi]$, then by the Cauchy–Schwarz inequality we have

$$|nb_n| \le |\alpha| + \sqrt{(n-1)\sum_{k=1}^{n-1} |S_k(e^{it})|^2}.$$

Finally, the estimate (2.4) of Theorem 2.1, which also remains true for $z = e^{it}$, $t \in [0, 2\pi]$, yields

$$|nb_n| \le |\alpha| + \sqrt{(n-1)(n-\alpha^2)}.$$

Hence, the proof is completed. □

Corollary 3.2. *If* $f \in \widehat{S}$ *and* f *is given by* (1.1), (1.3), *then*

$$|b_n| < 1$$

for $n = 2, 3, 4, \dots$.

Proof. The corollary follows immediately from Theorem 3.1. □

Consider a function $f = h + \overline{g}$ satisfying (1.2), (1.3) in Δ, such that

$$\Delta \ni z \mapsto \omega(z) := \alpha + (1-\alpha)z,$$

and

$$\Delta \ni z \mapsto h(z) := \frac{z}{1-z}.$$

The function f is well defined and so we can compute the coefficients of g as follows

$$b_n = 1 - \frac{1}{n} + \frac{\alpha}{n}$$

for $n = 2, 3, 4, \ldots$. Since $b_n \to 1$ as $n \to +\infty$, then it is clear, that the bound 1 in Corollary 3.2 can not be improved to be valid for all $n = 2, 3, 4, \ldots$.

Theorem 3.3. *If $f \in \widehat{S}^\alpha$, then*

$$|b_2| \leq \frac{1 + 2\alpha - \alpha^2}{2}. \tag{3.8}$$

The estimate can not be improved.

Proof. Let ω be the dilatation of f with the power series expansion (3.3). Consider the function

$$F(z) := \frac{\omega(z) - c_0}{1 - \overline{c_0}\omega(z)}, \quad z \in \Delta.$$

Recall, that $|\omega(z)| < 1$, $z \in \Delta$. Hence F satisfies the assumptions of the Schwarz lemma and by the inequality (2.2) we have $|F'(0)| \leq 1$, which gives

$$|c_1| = |\omega'(0)| \leq 1 - |c_0|^2. \tag{3.9}$$

On the other hand, the formula (3.6) from the proof of Theorem 3.1 gives

$$2|b_2| \leq |c_1| + 2|c_0|. \tag{3.10}$$

Now the estimate (3.8) follows immediately from (3.9) and the identity $|c_0| = |b_1| = \alpha$. The function f_ζ defined in Example 2.6 with $\zeta := \alpha$ shows that the inequality (3.8) can not be improved. □

Corollary 3.4. *If $f \in \widehat{S}$, then the estimate $|b_2| < 1$ can not be improved.*

Proof. Let α tend to 1 in the estimate (3.8) and the corollary follows from Theorem 3.3. □

Recall that the analytic part h of $f \in \widehat{S}^\alpha$ belongs to C. Hence, we have the following distortion estimate of h

$$\frac{1}{(1 + |z|)^2} \leq |h'(z)| \leq \frac{1}{(1 - |z|)^2}, \quad z \in \Delta. \tag{3.11}$$

Our next aim is to obtain the distortion estimate of g.

Theorem 3.5. *If $f \in \widehat{S}^\alpha$, then*

$$|g'(z)| \geq \frac{|\alpha - r|}{(1 - \alpha r)(1 + r)^2}, \quad z \in \Delta \tag{3.12}$$

and

$$|g'(z)| \leq \frac{\alpha + r}{(1 + \alpha r)(1 - r)^2}, \quad z \in \Delta, \tag{3.13}$$

where $r := |z|$. The estimates can not be improved.

Proof. Let ω of the form (3.3) be the dilatation of $f \in \widehat{S}^\alpha$ and $b_1 = c_0 = e^{i\phi}\alpha$ for some $\phi \in \mathbb{R}$. Consider the function

$$F(z) := \frac{e^{-i\phi}\omega(z) - \alpha}{1 - \alpha e^{-i\phi}\omega(z)}, \quad z \in \Delta.$$

It satisfies assumptions of the Schwarz lemma and using the estimate (2.1) we have $|F(z)| \le r$, which gives

$$\left| e^{-i\phi}\omega(z) - \alpha \right| \le r \left| \alpha e^{-i\phi}\omega(z) - 1 \right|.$$

This inequality is equivalent to

$$\left| e^{-i\phi}\omega(z) - \frac{\alpha(1-r^2)}{1-\alpha^2 r^2} \right| \le \frac{r(1-\alpha^2)}{1-\alpha^2 r^2} \tag{3.14}$$

and the equality holds only for the functions satisfying

$$\omega(z) = e^{i\phi}\frac{e^{i\psi}z + \alpha}{1 + \alpha e^{i\psi}z}, \quad z \in \Delta, \tag{3.15}$$

where $\psi \in \mathbb{R}$. From the formula (3.14) we obtain

$$|\omega(z)| = |e^{-i\phi}\omega(z)| \ge \left| \frac{\alpha(1-r^2)}{1-\alpha^2 r^2} - \frac{r(1-\alpha^2)}{1-\alpha^2 r^2} \right| = \frac{|\alpha - r|}{1 - \alpha r} \tag{3.16}$$

and

$$|\omega(z)| = |e^{-i\phi}\omega(z)| \le \frac{\alpha(1-r^2)}{1-\alpha^2 r^2} + \frac{r(1-\alpha^2)}{1-\alpha^2 r^2} = \frac{\alpha + r}{1 + \alpha r}. \tag{3.17}$$

Applying the estimate (3.11) together with (3.16) and (3.17) to the identity $g' = \omega h'$ we have (3.12) and (3.13), respectively. The function f_ζ defined in Example 2.6 with $\zeta := \alpha$ shows that the inequalities (3.12) and (3.13) can not be improved. □

Corollary 3.6. *If $f \in \widehat{S}$, then*

$$|g'(z)| \le \frac{1}{(1-r)^2}, \quad z \in \Delta, \tag{3.18}$$

where $r := |z|$. The estimate can not be improved.

Proof. Observe that the right-hand side of (3.13) increases in $[0, 1)$. Hence, let α tend to 1 in the estimate (3.13) and the corollary follows from Theorem 3.5. □

Let $f \in \widehat{S}^\alpha$. It is well known that the following growth estimate of $h \in C$ holds

$$\frac{|z|}{1+|z|} \le |h(z)| \le \frac{|z|}{1-|z|}, \quad z \in \Delta. \tag{3.19}$$

The growth estimate of g we derive, by integration, from the estimate on $|g'|$.

Theorem 3.7. *If $f \in \widehat{S}^\alpha$, then*

$$|g(z)| \le \frac{r}{1-r} + \frac{1-\alpha}{1+\alpha} \ln\left(\frac{1-r}{1+\alpha r}\right), \quad z \in \Delta, \tag{3.20}$$

where $r := |z|$. The estimate can not be improved.

Proof. Let $\gamma := [0, z]$. Applying the estimate (3.13) we have

$$|g(z)| = \left|\int_\gamma g'(\zeta)\,\mathrm{d}\zeta\right| \le \int_\gamma |g'(\zeta)|\,|\,\mathrm{d}\zeta\,| \le \int_0^r \frac{\alpha+\rho}{(1+\alpha\rho)(1-\rho)^2}\,\mathrm{d}\rho\,.$$

Integrating, we obtain the estimate (3.20). The function f_ζ defined in Example 2.6 with $\zeta := \alpha$ shows that the inequality (3.20) can not be improved. □

Corollary 3.8. *If $f \in \widehat{S}$, then*

$$|g(z)| \le \frac{r}{1-r}, \quad z \in \Delta, \tag{3.21}$$

where $r := |z|$. The estimate can not be improved.

Proof. Let α tend to 1 in the estimate (3.20), then the corollary follows from Theorem 3.7. □

Moreover, we give the growth estimate of f.

Theorem 3.9. *If $f \in \widehat{S}^\alpha$, then*

$$|f(z)| \ge \frac{2r}{1+r} - \frac{1+\alpha}{1-\alpha} \ln\left(\frac{1+r}{1+\alpha r}\right), \ z \in \Delta \tag{3.22}$$

and

$$|f(z)| \le \frac{2r}{1-r} + \frac{1-\alpha}{1+\alpha} \ln\left(\frac{1-r}{1+\alpha r}\right), \ z \in \Delta, \tag{3.23}$$

where $r := |z|$. The estimates can not be improved.

Proof. Let $z \in \Delta$. We denote $r := |z|$ and $m(r) := \inf\{|f(\zeta)| : |\zeta| = r\}$. Obviously $|f(z)| \ge m(r)$ and $\{w : |w| \le m(r)\} \subset f(\{\zeta : |\zeta| \le r\}) \subset f(\Delta)$. Hence, there exists z_r satisfying $|z_r| = r$ such that $m(r) = |f(z_r)|$. Let $\gamma(t) := tf(z_r)$, $t \in [0,1]$, then $\Gamma(t) := f^{-1}(\gamma(t))$, $t \in [0,1]$ is a well-defined Jordan arc and

$$|\Gamma(t)| \le s(t) := \int_0^t |\Gamma'(t)|\,\mathrm{d}\,t$$

for all $t \in [0,1]$. Since $f = h + \overline{g}$, then we can write

$$\begin{aligned} m(r) = |f(z_r)| &= \int_\gamma |\,\mathrm{d}\,w| = \int_\Gamma |\,\mathrm{d}\,f| = \int_\Gamma \left|h'(\zeta) + g'(\zeta)\frac{\mathrm{d}\overline{\zeta}}{\mathrm{d}\zeta}\right| |\,\mathrm{d}\zeta\,| \\ &\ge \int_\Gamma \big(|h'(\zeta)| - |g'(\zeta)|\big)\,|\,\mathrm{d}\zeta\,|. \end{aligned}$$

Observe, that for every $\zeta \in \Delta$ we have

$$|h'(\zeta)| - |g'(\zeta)| = |h'(\zeta)|(1 - |\omega(\zeta)|).$$

Applying the estimates (3.11) and (3.17) we obtain

$$|h'(\zeta)| - |g'(\zeta)| \geq \frac{1}{(1+|\zeta|)^2}\left(1 - \frac{\alpha + |\zeta|}{1+\alpha|\zeta|}\right) = \frac{(1-\alpha)(1-|\zeta|)}{(1+\alpha|\zeta|)(1+|\zeta|)^2}.$$

Hence, we can write

$$\begin{aligned} m(r) &\geq \int_\Gamma \left(\frac{(1-\alpha)(1-|\zeta|)}{(1+\alpha|\zeta|)(1+|\zeta|)^2}\right)|\,\mathrm{d}\zeta\,| \\ &\geq \int_0^1 \left(\frac{(1-\alpha)(1-|\Gamma(t)|)}{(1+\alpha|\Gamma(t)|)(1+|\Gamma(t)|)^2}\right)\mathrm{d}\,s(t) \\ &\geq \int_0^1 \left(\frac{(1-\alpha)(1-s(t))}{(1+\alpha s(t))(1+s(t))^2}\right)\mathrm{d}\,s(t) \\ &\geq \int_0^r \frac{(1-\alpha)(1-\rho)}{(1+\alpha\rho)(1+\rho)^2}\,\mathrm{d}\rho \\ &= \frac{2r}{1+r} - \frac{1+\alpha}{1-\alpha}\ln\left(\frac{1+r}{1+\alpha r}\right), \end{aligned}$$

which completes the proof of (3.22). To prove (3.23) we simply use the inequality

$$|f(z)| = |h(z) + \overline{g(z)}| \leq |h(z)| + |g(z)|.$$

Then, by applying (3.19) and (3.20) we have

$$|f(z)| \leq \frac{r}{1-r} + \frac{r}{1-r} + \frac{1-\alpha}{1+\alpha}\ln\left(\frac{1-r}{1+\alpha r}\right),$$

which completes the proof of (3.23). The function f_ζ defined in Example 2.6 with $\zeta := -\alpha$ and $\zeta := \alpha$ shows that the inequalities (3.22) and (3.23), respectively, can not be improved. □

Corollary 3.10. *If $f \in \widehat{S}$, then*

$$|f(z)| \leq \frac{2r}{1-r}, \quad z \in \Delta, \tag{3.24}$$

where $r := |z|$. The estimate can not be improved.

Proof. Let α tend to 1 in the estimate (3.23), then the corollary follows from Theorem 3.9. □

Finally, the growth estimate of $f \in \widehat{S}^\alpha$ yields a covering theorem.

Corollary 3.11. *If $f \in \widehat{S}^\alpha$, then*

$$\Delta(0, R) \subset f(\Delta),$$

where

$$R := 1 - \frac{1+\alpha}{1-\alpha} \ln\left(\frac{2}{1+\alpha}\right).$$

The constant R can not be improved.

Proof. If we let r tend to 1 in the estimate (3.22), then the corollary follows immediately from the argument principle for harmonic mappings (see [3]). The function f_ζ defined in Example 2.6 with $\zeta := -\alpha$ shows that the constant R can not be improved. □

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