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Generalized Weil functors on affine bundles

Abstract. We extend the construction by A. Weil onto affine bundles, and prove that all product preserving gauge bundle functors on affine bundles can be obtained by this extended construction.

0. Modern differential geometry clarifies that product preserving (gauge) bundles play very important roles. To such bundles one can lift many geometric objects as vector fields, forms, connections. To define such lifts only the product preserving property is used, see for ex. [1]. That is why, such bundles have been intensively studied and classified.

In the present paper we classify product preserving (gauge) bundles over affine bundles. Let us recall the following definitions (see for ex. [1]).

Let $F : \mathcal{AB} \to \mathcal{FM}$ be a covariant functor from the category $\mathcal{AB}$ of all affine bundles and their affine bundle homomorphisms into the category $\mathcal{FM}$ of fibred manifolds and their fibred maps. Let $B_{\mathcal{AB}} : \mathcal{AB} \to \mathcal{M}f$ and $B_{\mathcal{FM}} : \mathcal{FM} \to \mathcal{M}f$ be the respective base functors.

A gauge bundle functor on $\mathcal{AB}$ is a functor F satisfying $B_{\mathcal{FM}} \circ F = B_{\mathcal{AB}}$ and the localization condition: for every inclusion of an open affine subbundle $i_{E|U} : E|U \to E$, $F(E|U)$ is the restriction $p_E^{-1}(U)$ of $p_E : FE \to B_{\mathcal{AB}}(E)$ over U and $Fi_{E|U}$ is the inclusion $p_E^{-1}(U) \to FE$.

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Given two gauge bundle functors F_1, F_2 on $\mathcal{AB}$, by *a natural transformation* $\tau : F_1 \to F_2$ we shall mean a system of base preserving fibred maps $\tau_E : F_1 E \to F_2 E$ for every affine bundle E satisfying $F_2 f \circ \tau_E = \tau_G \circ F_1 f$ for every affine bundle homomorphism $f : E \to G$.

A gauge bundle functor F on $\mathcal{AB}$ is *product preserving* if for every product projections

$$E_1 \xleftarrow{pr_1} E_1 \times E_2 \xrightarrow{pr_2} E_2$$

in the category $\mathcal{AB}$,

$$FE_1 \xleftarrow{Fpr_1} F(E_1 \times E_2) \xrightarrow{Fpr_2} FE_2$$

are product projections in the category $\mathcal{FM}$. In other words $F(E_1 \times E_2) = F(E_1) \times F(E_2)$ modulo (Fpr_1, Fpr_2).

A simple example of such F is the functor $(\ )^{\to} : \mathcal{AB} \to \mathcal{FM}$ sending an affine bundle $E \to M$ into the corresponding vector bundle $E^{\to} \to M$ and any affine map $f : E \to G$ into the corresponding vector bundle map $f^{\to} : E^{\to} \to G^{\to}$. In fact $(\ )^{\to} : \mathcal{AB} \to \mathcal{VB}$, where $\mathcal{VB}$ is the category of vector bundles and their vector bundle maps.

Another example of such F is the tangent functor $T : \mathcal{AB} \to \mathcal{FM}$ sending an affine bundle $E \to M$ into $TE \to M$ and an affine bundle map $f : E \to G$ covering $\underline{f} : M \to N$ into the tangent map $Tf : TE \to TG$ over $\underline{f}$. More generally, by replacing T by other Weil functor T^A corresponding to a Weil algebra A we obtain the product preserving gauge bundle functor $T^A : \mathcal{AB} \to \mathcal{FM}$.

Another example is the vertical functor $V : \mathcal{AB} \to \mathcal{FM}$ sending an affine bundle $E \to M$ into its vertical bundle $VE = \bigcup_{z\in M} T(E_z) \to M$ and an affine bundle map $f : E \to G$ covering $\underline{f} : M \to N$ into the fibred map $Vf = \bigcup_{z\in M} T(f_z) : VE \to VG$ over $\underline{f}$. More generally, by replacing T by T^A we obtain the product preserving gauge bundle functor $V^A : \mathcal{AB} \to \mathcal{FM}$.

Functor $V^A : \mathcal{AB} \to \mathcal{FM}$ is the composition of the vertical Weil functor $V^A : \mathcal{FM} \to \mathcal{FM}$ with the forgetting functor $\mathcal{AB} \to \mathcal{FM}$. More generally, replacing $V^A : \mathcal{FM} \to \mathcal{FM}$ by the product preserving bundle functor T^μ on $\mathcal{FM}$ for some Weil algebra homomorphism $\mu : A \to B$, see [4], we obtain the product preserving gauge bundle functor $T^\mu : \mathcal{AB} \to \mathcal{FM}$.

Composing functor $(\ )^{\to} : \mathcal{AB} \to \mathcal{VB}$ with the product preserving gauge bundle functor $T^{A,V}$ on $\mathcal{VB}$ for some Weil algebra A and a Weil module V over A (i.e. A-module with $\dim_{\mathbb{R}}(V) < \infty$), see [4], we obtain new product preserving gauge bundle functor $T^{A,V} \circ (\ )^{\to}$ on $\mathcal{AB}$.

It will be shown that we can compose product preserving gauge bundle functors on $\mathcal{AB}$ and obtain product preserving gauge bundle functors on $\mathcal{AB}$.

In this paper modifying the method of [4] we generalize the construction of bundles of near A-points by A. Weil [5], and prove that all product preserving gauge bundle functors on $\mathcal{AB}$ can be obtained by this general construction.

Product preserving bundle functors on some other categories on manifolds have been described in [1]–[5].

All manifolds are assumed to be Hausdorff, finite dimensional, without boundaries and of class C^∞. All maps between manifolds are assumed to be of class C^∞.

1. Suppose we have a triple $(A, V, \mathbf{1})$, where $A = \mathbb{R} \oplus n_A$ is a Weil algebra, V is a Weil module over A and $\mathbf{1} \in V$ is an element. We generalize the construction of bundles of infinitely near points, [5].

Example 1. Given an affine bundle $E = (E \xrightarrow{p} M)$ let

$$T^{A,V,\mathbf{1}}E = \bigcup_{z\in M} \{(\varphi,\psi)|\ \varphi \in Hom(C^\infty_z(M), A),$$
$$\psi \in Hom_\varphi(FIBAFF_z(E), V),\ \psi(germ_z(1)) = \mathbf{1}\},$$

where $Hom(C^\infty_z(M), A)$ is the set of all unity preserving algebra homomorphisms φ from the algebra $C^\infty_z(M) = \{germ_z(g)|\ g : M \to \mathbb{R}\}$ into A and where $Hom_\varphi(FIBAFF_z(E), V)$ is the set of all module homomorphisms ψ over φ from the free $C^\infty_z(M)$-module $FIBAFF_z(E) = \{germ_z(h) \mid h : E \to \mathbb{R}$ is fiber affine$\}$ into V. Then $T^{A,V,\mathbf{1}}E$ is a fibred manifold over M. A local affine bundle trivialization $(x^1 \circ p, \dots, x^m \circ p, y^1, \dots, y^k) : E|U \tilde{=} \mathbb{R}^m \times \mathbb{R}^k$ on E induces a fiber bundle trivialization $(\tilde{x}^1, \dots, \tilde{x}^m, \tilde{y}^1, \dots, \tilde{y}^k) : T^{A,V,\mathbf{1}}E|U \tilde{=} A^m \times V^n = \mathbb{R}^m \times n_A^m \times V^n$ by $\tilde{x}^i(\varphi,\psi) = \varphi(germ_z(x^i)) \in A$, $\tilde{y}^j(\varphi,\psi) = \psi(germ_z(y^j)) \in V$, $(\varphi,\psi) \in T^{A,V,\mathbf{1}}_z E$, $z \in U$, $i = 1, \dots, m$, $j = 1, \dots, k$. Given another affine bundle $G = (G \xrightarrow{q} N)$ and an affine bundle homomorphism $f : E \to G$ over $\underline{f} : M \to N$ let $T^{A,V,\mathbf{1}}f : T^{A,V,\mathbf{1}}E \to T^{A,V,\mathbf{1}}G$,

$$T^{A,V,\mathbf{1}}f(\varphi,\psi) = (\varphi \circ \underline{f}^*_z, \psi \circ f^*_z),$$

$(\varphi,\psi) \in T^{A,V,\mathbf{1}}_z E$, $z \in M$, where the mappings $\underline{f}^*_z : C^\infty_{\underline{f}(z)}(N) \to C^\infty_z(M)$ and $f^*_z : FIBAFF_{\underline{f}(z)}(G) \to FIBAFF_z(E)$ are given by the pull-back with respect to $\underline{f}$ and f. Then $T^{A,V,\mathbf{1}}f$ is a fibred map over $\underline{f}$.

Clearly, $T^{A,V,\mathbf{1}}$ is a product preserving gauge bundle functor on $\mathcal{AB}$. It is called the product preserving gauge bundle functor on $\mathcal{AB}$ corresponding to the triple $(A, V, \mathbf{1})$.

Proposition 1. (i) *Given an affine bundle* $p : E \to M$*,*

$$T^{A,V,\mathbf{1}}p : T^{A,V,\mathbf{1}}E \to T^A M = T^{A,V,\mathbf{1}}M$$

is the affine bundle with the corresponding vector bundle $T^{A,V,0}p : T^{A,V,0}E \to T^A M = T^{A,V,0}M$*, where* M *is treated as the trivial affine bundle* $id_M : M \to M$ *and* $p : E \to M$ *is treated as the trivial affine bundle map covering* id_M*.*

(ii) *Given an affine bundle morphism* $f : E \to G$ *covering* $\underline{f} : M \to N$*,*

$$T^{A,V,\mathbf{1}}f : T^{A,V,\mathbf{1}}E \to T^{A,V,\mathbf{1}}G$$

is an affine bundle map covering $T^A\underline{f} : T^A M \to T^A N$ *with the corresponding vector bundle map* $T^{A,V,0}f : T^{A,V,0}E \to T^{A,V,0}G$*.*

(iii) *Vector bundle* $T^{A,V,0}E \to T^A M$ *is canonically isomorphic to* $T^{A,V}E^{\to} \to T^A M$ *(see* [4] *for* $T^{A,V}$*) by some vector bundle isomorphism covering the identity of* $T^A M$*.*

Proof. Parts (i) and (ii) are simple observations. More precisely, given $(\varphi, \psi_1), (\varphi, \psi_2) \in T^{A,V,0}E$ and $\alpha \in \mathbb{R}$ we put $(\varphi, \psi_1) + (\varphi, \psi_2) := (\varphi, \psi_1 + \psi_2) \in T^{A,V,0}E$ and $\alpha(\varphi, \psi_1) := (\varphi, \alpha\psi_1) \in T^{A,V,0}E$. That is why, $T^{A,V,0}E \to T^A M$ is a vector bundle. Similarly, given $(\varphi, \psi_1) \in T^{A,V,\mathbf{1}}E$ and $(\varphi, \psi_2) \in T^{A,V,0}E$ we put $(\varphi, \psi_1) + (\varphi, \psi_2) := (\varphi, \psi_1 + \psi_2) \in T^{A,V,\mathbf{1}}E$. That is why, $T^{A,V,\mathbf{1}}E \to T^A M$ is an affine bundle with the corresponding vector bundle $T^{A,V,0}E \to T^A M$.

Part (iii) will be clear after Section 8 because of $T^{A,V} \circ (\ )^{\to}$ and $T^{A,V,0}$ have isomorphic the corresponding triples. □

Remark 1. Let us note that in Example 1 we do not assume that V is free. For example, the triple $(A, n_A, \mathbf{1})$ is in question.

2. Suppose we have a product preserving gauge bundle functor F on $\mathcal{AB}$.

Example 2. (i) Let $A^F = (G^F\mathbb{R}, G^F(+), G^F(\cdot), G^F(0), G^F(1))$, where $G^F : \mathcal{M}f \to \mathcal{FM}$, $G^F M = F(M \xrightarrow{id_M} M)$, $G^F f = Ff : G^F M \to G^F N$, and where $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the sum map, $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the multiplication map, $0 : \mathbb{R} \to \mathbb{R}$ is the zero and $1 : \mathbb{R} \to \mathbb{R}$ is the unity. Then A^F is a Weil algebra.

(ii) Let $V^F = (F(\mathbb{R} \to pt), F(+), F(\cdot), F(0))$, where pt is the one point manifold, $\mathbb{R} \to pt$ is the affine bundle with the corresponding vector bundle $\mathbb{R} \to pt$, $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the sum map being an affine bundle homomorphism $(\mathbb{R} \to pt) \times (\mathbb{R} \to pt) \to (\mathbb{R} \to pt)$ over $pt \times pt \to pt$, $0 : \mathbb{R} \to \mathbb{R}$ is the zero map being an affine bundle homomorphism $(\mathbb{R} \to pt) \to (\mathbb{R} \to pt)$ over $pt \to pt$ and $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the multiplication map being an affine bundle homomorphism $(\mathbb{R} \xrightarrow{id_{\mathbb{R}}} \mathbb{R}) \times (\mathbb{R} \to pt) \to (\mathbb{R} \to pt)$ over $\mathbb{R} \times pt \to pt$. Then V^F is a Weil module over A^F.

(iii) Let $\mathbf{1}^F \in V^F$ be the unique element from the image of $F1 : V^F \to V^F$, where $1 : \mathbb{R} \to \mathbb{R}$ is the constant map being an affine bundle homomorphism $(\mathbb{R} \to pt) \to (\mathbb{R} \to pt)$.

The triple $(A^F, V^F, \mathbf{1}^F)$ is called the triple corresponding to F.

For example, the triple corresponding to $T^A : \mathcal{AB} \to \mathcal{FM}$ is $(A, A, 1)$, where A is the A-module in obvious way and $1 \in A$ is the unity of Weil algebra A. The triple corresponding to $V^A : \mathcal{AB} \to \mathcal{FM}$ is $(\mathbb{R}, A, 1)$. The triple corresponding to $T^\mu : \mathcal{AB} \to \mathcal{FM}$ for some Weil algebra homomorphism $\mu : A \to B$ is $(A, B, 1)$, where B is the A-module by μ and $1 \in B$ is the unity of Weil algebra B. The triple corresponding to $(\ )^{\to} : \mathcal{AB} \to \mathcal{FM}$ is $(\mathbb{R}, \mathbb{R}, 0)$. The triple corresponding to $T^{A,V} \circ (\ )^{\to}$ is isomorphic to $(A, V, 0)$.

3. Let F be a product preserving gauge bundle functor on $\mathcal{AB}$ and let $(A^F, V^F, \mathbf{1}^F)$ be its corresponding triple. Let $T^{A^F,V^F,\mathbf{1}^F}$ be the product preserving gauge bundle functor on $\mathcal{AB}$ corresponding to $(A^F, V^F, \mathbf{1}^F)$. We prove $F \tilde{=} T^{A^F,V^F,\mathbf{1}^F}$.

For every affine bundle $E = (E \xrightarrow{p} M)$ we construct a fibred map $\Theta_E : FE \to T^{A^F,V^F,\mathbf{1}^F} E$ covering id_M as follows. If $y \in F_z E$, $z \in M$, we define $\varphi_y : C^\infty_z(M) \to A^F$, $\varphi_y(germ_z(g)) = F(g \circ p)(y) \in A^F = F(\mathbb{R} \xrightarrow{id_\mathbb{R}} \mathbb{R})$, $g : M \to \mathbb{R}$, where $g \circ p : E \to \mathbb{R}$ is considered as the affine bundle homomorphism $(E \xrightarrow{p} M) \to (\mathbb{R} \xrightarrow{id_\mathbb{R}} \mathbb{R})$ over $g : M \to \mathbb{R}$. Then φ_y is an algebra homomorphism. If $y \in F_z E$, $z \in M$, we define $\psi_y : FIBAFF_z(E) \to V^F$, $\psi_y(germ_z(f)) = F(f)(y)$, $f : E \to \mathbb{R}$ is fibre affine, where f is considered as the affine bundle map $(E \xrightarrow{p} M) \to (\mathbb{R} \to pt)$ over $M \to pt$. Then ψ_y is a module homomorphism over φ_y and $\psi_y(germ_z(1)) = \mathbf{1}^F$. We put $\Theta_E(y) = (\varphi_y, \psi_y) \in T_z^{A^F,V^F,\mathbf{1}^F} E$, $y \in F_z E$, $z \in M$.

Proposition 2. $\Theta : F \to T^{A^F,V^F,\mathbf{1}^F}$ *is a natural isomorphism.*

Proof. It is sufficient to show that Θ_E is a diffeomorphism for any affine bundle E. Applying affine bundle trivialization, we can assume that $E = \mathbb{R}^m \times \mathbb{R}^k$ is the trivial affine bundle over $\mathbb{R}^m$ with the corresponding trivial vector bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. Since F and $T^{A^F,V^F,\mathbf{1}^F}$ are product preserving and E is a (multi) product of $\mathbb{R} \xrightarrow{id_\mathbb{R}} \mathbb{R}$ and $\mathbb{R} \to pt$, we can assume that E is either $\mathbb{R} \xrightarrow{id_\mathbb{R}} \mathbb{R}$ or $\mathbb{R} \to pt$.

(I) $E = (\mathbb{R} \xrightarrow{id_\mathbb{R}} \mathbb{R})$. Consider $G^F\mathbb{R} \xrightarrow{\Theta_E} T^{A^F,V^F,\mathbf{1}^F}(\mathbb{R} \xrightarrow{id_\mathbb{R}} \mathbb{R}) \xrightarrow{\tilde{x}^1} A^F$, where $\tilde{x}^1$ is induced by $x^1 = id_\mathbb{R} : \mathbb{R} \to \mathbb{R}$, see Example 1. This composition is the identity map $G^F\mathbb{R} = A^F$. Hence Θ_E is a diffeomorphism.

(II) $E = (\mathbb{R} \to pt)$. Consider $F(\mathbb{R} \to pt) \xrightarrow{\Theta_E} T^{A^F,V^F,\mathbf{1}^F}(\mathbb{R} \to pt) \xrightarrow{\tilde{y}^1} V^F$, where $\tilde{y}^1$ is induced by $y^1 = id_\mathbb{R} : \mathbb{R} \to \mathbb{R}$. This composition is the identity map $F(\mathbb{R} \to pt) = V^F$. Hence Θ_E is a diffeomorphism. □

From Propositions 1 and 2 we obtain.

Proposition 3. *Any product preserving gauge bundle functor F on $\mathcal{AB}$ has values in $\mathcal{AB}$. More precisely, given an affine bundle $p : E \to M$, $Fp : FE \to FM$ is the affine bundle (by the isomorphism Θ from Proposition 2),* *and given an affine bundle map $f : E \to G$ covering $\underline{f} : M \to N$, $Ff : FE \to FG$ is an affine bundle map covering $F\underline{f} : FM \to FN$.*

4. Let $(A, V, \mathbf{1})$ be a triple, where A is a Weil algebra, V is a Weil module over A and $\mathbf{1} \in V$ is an element. Let $T^{A,V,\mathbf{1}}$ be the corresponding gauge bundle functor on $\mathcal{AB}$. Let $(\tilde{A}, \tilde{V}, \tilde{\mathbf{1}})$ be the triple corresponding to $T^{A,V,\mathbf{1}}$.

Proposition 4. $(A, V, \mathbf{1}) \tilde{=} (\tilde{A}, \tilde{V}, \tilde{\mathbf{1}})$.

Proof. Clearly, $\tilde{A} = T^{A,V,\mathbf{1}}(\mathbb{R} \xrightarrow{id_{\mathbb{R}}} \mathbb{R})$ and $\tilde{V} = T^{A,V,\mathbf{1}}(\mathbb{R} \to pt)$. Let $\mathcal{O} = \tilde{x}^1 : T^{A,V,\mathbf{1}}(\mathbb{R} \xrightarrow{id_{\mathbb{R}}} \mathbb{R}) \to A$ and $\Pi = \tilde{y}^1 : T^{A,V,\mathbf{1}}(\mathbb{R} \to pt) \to V$, where $\tilde{x}^1$ is induced by $x^1 = id_{\mathbb{R}}$ and $\tilde{y}^1$ is induced by $y^1 = id_{\mathbb{R}}$, see Example 1. Then $\mathcal{O} : \tilde{A} \to A$ is an algebra isomorphism, $\Pi : \tilde{V} \to V$ is a module isomorphism over $\mathcal{O}$ and $\Pi(\tilde{\mathbf{1}}) = \mathbf{1}$. □

5. Let $(A_1, V_1, \mathbf{1}_1)$ and $(A_2, V_2, \mathbf{1}_2)$ be triples, where A_i is a Weil algebra, V_i is a Weil module over A_i and $\mathbf{1}_i \in V_i$ is an element, $i = 1, 2$. Let (μ, ν) be a morphism from $(A_1, V_1, \mathbf{1}_1)$ into $(A_2, V_2, \mathbf{1}_2)$, i.e. $\mu : A_1 \to A_2$ is an algebra homomorphism, $\nu : V_1 \to V_2$ is a module homomorphism over μ and $\nu(\mathbf{1}_1) = \mathbf{1}_2$.

Example 3. Let $E \to M$ be an affine bundle. We define $\tau^{\mu,\nu}_E : T^{A_1,V_1,\mathbf{1}_1}E \to T^{A_2,V_2,\mathbf{1}_2}E$, $\tau^{\mu,\nu}_E(\varphi, \psi) = (\mu \circ \varphi, \nu \circ \psi)$, $(\varphi, \psi) \in T^{A_1,V_1,\mathbf{1}_1}_z E$, $z \in M$. Then $\tau^{\mu,\nu} : T^{A_1,V_1,\mathbf{1}_1} \to T^{A_2,V_2,\mathbf{1}_2}$ is a natural transformation.

6. Let $\tau : F_1 \to F_2$ be a natural transformation between product preserving gauge bundle functors on $\mathcal{AB}$. Let $(A^{F_1}, V^{F_1}, \mathbf{1}^{F_1})$ and $(A^{F_2}, V^{F_2}, \mathbf{1}^{F_2})$ be the triples corresponding to F_1 and F_2.

Example 4. Let $\mu^\tau := \tau_{id_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}} : A^{F_1} \to A^{F_2}$ and $\nu^\tau := \tau_{\mathbb{R}\to pt} : V^{F_1} \to V^{F_2}$. Then (μ^τ, ν^τ) is a morphism of triples corresponding to F_1 and F_2.

7. We are now in position to prove the following theorem.

Theorem 1. *The correspondence "$F \to (A^F, V^F, \mathbf{1}^F)$" induces a bijective correspondence between the equivalence classes of product preserving gauge bundle functors F on $\mathcal{AB}$ and the equivalence classes of triples $(A, V, \mathbf{1})$ consisting of a Weil algebra A, a Weil module V over A and an element $\mathbf{1} \in V$. The inverse correspondence is induced by the correspondence "$(A, V, \mathbf{1}) \to T^{A,V,\mathbf{1}}$".*

Proof. The correspondence "$[F] \to [(A^F, V^F, \mathbf{1}^F)]$" is well defined. For, if $\tau : F_1 \to F_2$ is an isomorphism, then so is $(\mu^\tau, \nu^\tau) : (A^{F_1}, V^{F_1}, \mathbf{1}^{F_1}) \to (A^{F_2}, V^{F_2}, \mathbf{1}^{F_2})$.

The correspondence "$[(A, V, \mathbf{1})] \to [T^{A,V,\mathbf{1}}]$" is well defined. For, if $(\mu, \nu) : (A_1, V_1, \mathbf{1}_1) \to (A_2, V_2, \mathbf{1}_2)$ is an isomorphism, then so is $\tau^{\mu,\nu} : T^{A_1,V_1,\mathbf{1}_1} \to T^{A_2,V_2,\mathbf{1}_2}$.

From Proposition 2 it follows that $[F] = [T^{A^F,V^F,\mathbf{1}^F}]$. From Proposition 4 it follows that $[(A,V,\mathbf{1})] = [(A^F,V^F,\mathbf{1}^F)]$ if $F = T^{A,V,\mathbf{1}}$. □

8. Let F_1 and F_2 be two product preserving gauge bundle functors on $\mathcal{AB}$. Let $(A^{F_1},V^{F_1},\mathbf{1}^{F_1})$ and $(A^{F_2},V^{F_2},\mathbf{1}^{F_2})$ be the corresponding triples.

Proposition 5. *Let* $(\mu,\nu) : (A^{F_1},V^{F_1},\mathbf{1}^{F_1}) \to (A^{F_2},V^{F_2},\mathbf{1}^{F_2})$ *be a morphism of triples. Let* $\tau^{[\mu,\nu]} : F_1 \to F_2$ *be a natural transformation given by the composition* $F_1 \xrightarrow{\Theta} T^{A^{F_1},V^{F_1},\mathbf{1}^{F_1}} \xrightarrow{\tau^{\mu,\nu}} T^{A^{F_2},V^{F_2},\mathbf{1}^{F_2}} \xrightarrow{\Theta^{-1}} F_2$*, where* Θ *is as in Proposition 2 and* $\tau^{\mu,\nu}$ *is described in Example 3. Then* $\tau = \tau^{[\mu,\nu]}$ *is the unique natural transformation* $F_1 \to F_2$ *such that* $(\mu^\tau,\nu^\tau) = (\mu,\nu)$*, where* (μ^τ,ν^τ) *is as in Example 4.*

Proof. First we prove the uniqueness part. Suppose $\overline{\tau} : F_1 \to F_2$ is another natural transformation such that $(\mu^{\overline{\tau}},\nu^{\overline{\tau}}) = (\mu,\nu)$. Then $\overline{\tau}$ coincides with τ on affine bundles $\mathbb{R} \xrightarrow{id_{\mathbb{R}}} \mathbb{R}$ and $\mathbb{R} \to pt$ because of the definition of (μ^τ,ν^τ). Hence $\overline{\tau} = \tau$ because of the same argument as in the proof of Proposition 2.

The existence part follows from the easy to verify equalities $\Theta^{-1}_{\mathbb{R}\to pt} \circ \tau^{\mu,\nu}_{\mathbb{R}\to pt} \circ \Theta_{\mathbb{R}\to pt} = \nu$ and $\Theta^{-1}_{id_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}} \circ \tau^{\mu,\nu}_{id_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}} \circ \Theta_{id_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}} = \mu$. □

Now, the following theorem is clear.

Theorem 2. *Let* F_1 *and* F_2 *be two product preserving gauge bundle functors on* $\mathcal{AB}$*. The correspondence "*$\tau \to (\mu^\tau,\nu^\tau)$*" is a bijection between the natural transformations* $F_1 \to F_2$ *and the morphisms* $(A^{F_1},V^{F_1},\mathbf{1}^{F_1}) \to (A^{F_2},V^{F_2},\mathbf{1}^{F_2})$ *between corresponding triples. The inverse correspondence is "*$(\mu,\nu) \to \tau^{[\mu,\nu]}$*".*

9. Using Proposition 3 one can define the composition $F_2 \circ F_1$ of product preserving gauge bundle functors F_1 and F_2 on $\mathcal{AB}$.

Example 5. Let $p : E \to M$ be an affine bundle. Then $F_1 p : F_1 E \to F_1 M$ is the affine bundle (Proposition 3). Applying F_2, we define a fibred manifold $F_2 \circ F_1(E) := F_2(F_1 E \xrightarrow{F_1 p} F_1 M)$ over M, where the projection $F_2 \circ F_1(E) \to M$ is the composition $F_2 \circ F_1(E) \to F_1 M \to M$ of projections for F_2 and F_1. Let $f : E \to G$ be an affine bundle homomorphism covering $\underline{f} : M \to N$. Then $F_1 f : F_1 E \to F_2 E$ is an affine bundle homomorphism over $F_1 \underline{f}$ (Proposition 3). We put $F_2 \circ F_1(f) := F_2(F_1 f) : F_2 \circ F_1(E) \to F_2 \circ F_1(G)$ and get a fibred map covering $\underline{f}$. $F_2 \circ F_1$ is a product preserving gauge bundle functor on $\mathcal{AB}$.

10. Let us compute the triple $(A^{F_2\circ F_1},V^{F_2\circ F_1},\mathbf{1}^{F_2\circ F_1})$ corresponding to the composition $F_2 \circ F_1$ of product preserving gauge bundle functors F_1 and F_2 on $\mathcal{AB}$.

By tensoring A^{F_1} and A^{F_2} we obtain the Weil algebra $A^{F_1} \otimes_{\mathbb{R}} A^{F_2}$. By tensoring V^{F_1} and V^{F_2} we obtain the module $V^{F_1} \otimes_{\mathbb{R}} V^{F_2}$ over $A^{F_1} \otimes_{\mathbb{R}} A^{F_2}$. We have also $\mathbf{1}^{F_1} \otimes \mathbf{1}^{F_2} \in V^{F_1} \otimes_{\mathbb{R}} V^{F_2}$.

Proposition 6.

$$\big(A^{F_2\circ F_1}, V^{F_2\circ F_1}, \mathbf{1}^{F_2\circ F_1}\big) \tilde{=} \big(A^{F_1}\otimes_{\mathbb{R}} A^{F_2}, V^{F_1}\otimes_{\mathbb{R}} V^{F_2}, \mathbf{1}^{F_1}\otimes \mathbf{1}^{F_2}\big).$$

Proof. We have to construct an algebra isomorphism $\tilde{\mu} : A^{F_1} \otimes_{\mathbb{R}} A^{F_2} \to A^{F_2\circ F_1}$ and a module isomorphism $\tilde{\nu} : V^{F_1} \otimes_{\mathbb{R}} V^{F_2} \to V^{F_2\circ F_1}$ over $\tilde{\mu}$ such that $\tilde{\nu}(\mathbf{1}^{F_1}\otimes \mathbf{1}^{F_2}) = \mathbf{1}^{F_2\circ F_1}$.

For any point $a \in A^{F_1}$ the map $i_a : \mathbb{R} \to A^{F_1}$, $i_a(t) = ta$, $t \in \mathbb{R}$ is a homomorphism between affine bundles $id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ and $id_{A^{F_1}} : A^{F_1} \to A^{F_1}$. Applying F_2, we obtain $F_2(i_a) : A^{F_2} \to A^{F_2\circ F_1}$. Define $\tilde{\mu} : A^{F_1} \times A^{F_2} \to A^{F_2\circ F_1}$, $\tilde{\mu}(a, b) = F_2(i_a)(b)$, $a \in A^{F_1}$, $b \in A^{F_2}$. Using the definitions of the algebra operations, one can show that $\tilde{\mu}$ is $\mathbb{R}$-bilinear. Then (by the universal factorization property) we have a linear map $\tilde{\mu} : A^{F_1} \otimes_{\mathbb{R}} A^{F_2} \to A^{F_2\circ F_1}$, $\tilde{\mu}(a \otimes b) = F_2(i_a)(b)$, $a \in A^{F_1}$, $b \in A^{F_2}$. Considering bases (over $\mathbb{R}$) of A^{F_1} and A^{F_2} and using the product property for F_2, one can prove that $\tilde{\mu}$ is an isomorphism. Using the definitions of the algebra operations, one can show that $\tilde{\mu}$ is an algebra isomorphism.

For any point $u \in V^{F_1}$ the map $i_u : \mathbb{R} \to V^{F_1}$, $i_u(t) = tu$, $t \in \mathbb{R}$ is a homomorphism between affine bundles $\mathbb{R} \to pt$ and $V^{F_1} \to pt$. Applying F_2, we obtain $F_2(i_u) : V^{F_2} \to V^{F_2\circ F_1}$. Define $\tilde{\nu} : V^{F_1} \times V^{F_2} \to V^{F_2\circ F_1}$, $\tilde{\nu}(u, w) = F_2(i_u)(w)$, $u \in V^{F_1}$, $w \in V^{F_2}$. Similarly as $\tilde{\mu}$, $\tilde{\nu}$ is also $\mathbb{R}$-bilinear. Then we have a linear map $\tilde{\nu} : V^{F_1} \otimes_{\mathbb{R}} V^{F_2} \to V^{F_2\circ F_1}$, $\tilde{\nu}(u \otimes w) = F_2(i_u)(w)$, $u \in V^{F_1}$, $w \in V^{F_2}$. Similarly as $\tilde{\mu}$, $\tilde{\nu}$ is a linear isomorphism. Using the definitions of the module operations, one can show that $\tilde{\nu}$ is a module isomorphism over $\tilde{\mu}$.

Next, using the definition of the fixed elements it is easy to see that $\tilde{\nu}(\mathbf{1}^{F_1}\otimes \mathbf{1}^{F_2}) = \mathbf{1}^{F_2\circ F_1}$ □

Proposition 7. $F_2 \circ F_1 \tilde{=} F_1 \circ F_2$.

Proof. The exchange isomorphism

$$\big(A^{F_1}\otimes_{\mathbb{R}} A^{F_2}, V^{F_1}\otimes_{\mathbb{R}} V^{F_2}, \mathbf{1}^{F_1}\otimes \mathbf{1}^{F_2}\big) \tilde{=} \big(A^{F_2}\otimes_{\mathbb{R}} A^{F_1}, V^{F_2}\otimes_{\mathbb{R}} V^{F_1}, \mathbf{1}^{F_2}\otimes \mathbf{1}^{F_1}\big)$$

induces the natural isomorphism $F_2 \circ F_1 \tilde{=} F_1 \circ F_2$. □

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