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Second order nonholonomic connections from second order nonholonomic ones

Abstract. We describe all $\mathcal{FM}_{m,n}$-natural operators $A : \tilde{J}^2 \rightsquigarrow \tilde{J}^2$ transforming second order nonholonomic connections $\Theta : Y \to \tilde{J}^2 Y$ on fibred manifolds $Y \to M$ into second order nonholonomic connections $A(\Theta) : Y \to \tilde{J}^2 Y$ on $Y \to M$.

Manifolds and maps are assumed to be of class C^∞. Manifolds are assumed to be finite dimensional and without boundaries.

Let $\mathcal{FM}$ be the category of fibred manifolds and their fibred maps, let $\mathcal{FM}_m$ be the category of fibred manifolds with m-dimensional bases and their fibred maps covering embeddings, and let $\mathcal{FM}_{m,n}$ be the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred embeddings.

Given a fibred manifold $Y \to M$ we have its jet prolongation J^1Y (the bundle of 1-jets $j^1_x\sigma$ of sections of $Y \to M$) and given an $\mathcal{FM}_m$-map $f : Y_1 \to Y_2$ covering $\underline{f} : M_1 \to M_2$ we have a fibred map $J^1 f : J^1Y_1 \to J^1Y_2$ covering f given by $J^1 f(j^1_x\sigma) = j^1_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})$, $j^1_x\sigma \in J^1Y_1$. The functor $J^1 : \mathcal{FM}_m \to \mathcal{FM}$ is a (fiber product preserving) bundle functor in the sense of [2]. Iterating J^1 we obtain the second order nonholonomic jet (fiber product preserving) bundle functor $\tilde{J}^2 := J^1J^1 : \mathcal{FM}_m \to \mathcal{FM}$ ($\tilde{J}^2(Y \to M) = J^1(J^1Y \to M)$).

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A first order connection on a fibred manifold $Y \to M$ is a section $\Gamma : Y \to J^1Y$ of $J^1Y \to Y$. A second order nonholonomic connection on a fibred manifold $Y \to M$ is a section $\Theta : Y \to \tilde{J}^2Y$ of $\tilde{J}^2Y \to Y$.

Proposition 1 ([1])**.** *Second order nonholonomic connections* Θ *on* $Y \to M$ *are in bijection with couples* (Γ_1, Γ_2, G) *consisting of first order connections* Γ_1, Γ_2 *on* $Y \to M$ *and tensor fields* $G : Y \to \otimes^2 T^*M \otimes VY$.

Let Γ_1, Γ_2 be first order connections on $Y \to M$. Let $Q = \Gamma_1 - \Gamma_2 : Y \to T^*M \otimes VY$ be the "difference" tensor field, where the operation "$-$" is the difference in the affine bundle $J^1Y \to Y$ with the corresponding vector bundle $T^*M \otimes VY$ over Y. Then Proposition 1 can be reformulated as follows.

Proposition 1′. *Second order nonholonomic connections* Θ *on* $Y \to M$ *are in bijection with couples* (Γ, Q, G) *consisting of first order connections* Γ *on* $Y \to M$ *and tensor fields* $Q : Y \to T^*M \otimes VY$ *and* $G : Y \to \otimes^2 T^*M \otimes VY$.

In the present paper we study the problem how a second order nonholonomic connection $\Theta : Y \to \tilde{J}^2Y$ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ can induce canonically a second order nonholonomic connection $A(\Theta) : Y \to \tilde{J}^2Y$ on $Y \to M$. This problem is reflected in the concept of $\mathcal{FM}_{m,n}$-natural operators $A : \tilde{J}^2 \rightsquigarrow \tilde{J}^2$. In the present note we find all $\mathcal{FM}_{m,n}$-natural operators A in question.

We remark that a general concept of natural operators can be found in [2]. In the present note we need (in particular) the following partial case of natural operators.

A $\mathcal{FM}_{m,n}$-natural operator $A : \tilde{J}^2 \rightsquigarrow \tilde{J}^2$ is a system of $\mathcal{FM}_{m,n}$-invariant regular operators (functions)

$$A = A_{Y\to M} : \Gamma(\tilde{J}^2Y) \to \Gamma(\tilde{J}^2Y)$$

for any $\mathcal{FM}_{m,n}$-object $Y \to M$, where $\Gamma(\tilde{J}^2Y)$ is the set of second order nonholonomic connections on $Y \to M$. The invariance means that if $\Theta_1 \in \Gamma(\tilde{J}^2Y_1)$ and $\Theta_2 \in \Gamma(\tilde{J}^2Y_2)$ are f-related by an $\mathcal{FM}_{m,n}$-map $f : Y_1 \to Y_2$ (i.e. $\tilde{J}^2f \circ \Theta_1 = \Theta_2 \circ f$) then $A(\Theta_1)$ and $A(\Theta_2)$ are f-related. The regularity means that A transforms smoothly parametrized families of second order nonholonomic connections into smoothly parametrized ones.

According to Proposition 1 it is sufficient to classify all $\mathcal{FM}_{m,n}$-natural operators $A_1 : \tilde{J}^2 \rightsquigarrow J^1$ transforming second order nonholonomic connections Θ on $Y \to M$ into first order connections $A_1(\Theta)$ on $Y \to M$ and to classify all $\mathcal{FM}_{m,n}$-natural operators $A_2 : \tilde{J}^2 \rightsquigarrow T^*B \otimes V$ transforming second order nonholonomic connections Θ on $Y \to M$ into tensor fields $A_2(\Theta) : Y \to T^*M \otimes VY$ and to classify all $\mathcal{FM}_{m,n}$-natural operators $A_3 : \tilde{J}^2 \rightsquigarrow \otimes^2 T^*B \otimes V$ transforming second order nonholonomic connections Θ on $Y \to M$ into tensor fields $A_3(\Theta) : Y \to \otimes^2 T^*M \otimes VY$ (the definitions of the above type natural operators are quite similar to the definition of natural operators $\tilde{J}^2 \rightsquigarrow \tilde{J}^2$).

At first we prove

Proposition 2. *Any $\mathcal{FM}_{m,n}$-natural operator $A_2 : \tilde{J}^2 \rightsquigarrow T^*B \otimes V$ is of the form*

$$A_2(\Theta) = \tau Q$$

for some $\tau \in \mathbf{R}$, where $\Theta = (\Gamma, Q, G)$ is an arbitrary second order nonholonomic connection on $Y \to M$.

Proof. Since a $\mathcal{FM}_{m,n}$-map $(x, y - \sigma(x))$ sends $j^1_0(x, \sigma(x))$ into $j^1_0(x, 0)$, so $J^1_0(\mathbf{R}^m \times \mathbf{R}^n)$ is the $\mathcal{FM}_{m,n}$-orbit of $\theta^o = j^1_0(x, 0) \in J^1_0(\mathbf{R}^m \times \mathbf{R}^n)$. Then (by the $\mathcal{FM}_{m,n}$-invariance of A_2) A_2 is determined by the values

$$A_2(\Gamma, Q, G)(0, 0) \in T^*_0\mathbf{R}^m \otimes V_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \tag{1}$$

for all first order connections Γ on $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ with $\Gamma(0, 0) = \theta^o$, all tensor fields $Q : \mathbf{R}^m \times \mathbf{R}^n \to T^*\mathbf{R}^m \otimes V(\mathbf{R}^m \times \mathbf{R}^n)$ and all tensor fields $G : \mathbf{R}^m \times \mathbf{R}^n \to \otimes^2 T^*\mathbf{R}^m \otimes V(\mathbf{R}^m \times \mathbf{R}^n)$. Then using the invariance of A_2 with respect to the homotheties $\frac{1}{t} id_{\mathbf{R}^m \times \mathbf{R}^n}$ for $t > 0$ and putting $t \to 0$ we deduce that A_2 is determined by the value

$$A_2(\Gamma^o, Q^o, 0)(0) \in T^*_0\mathbf{R}^m \otimes V_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \tag{2}$$

where Γ^o is the trivial first order connection on $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$, and $Q^o : \mathbf{R}^m \times \mathbf{R}^n \to T^*\mathbf{R}^m \otimes V(\mathbf{R}^m \times \mathbf{R}^n)$ is the "constant" tensor field such that $Q^o(0, 0) = Q(0, 0)$. Then using the invariance of $A_2(\Gamma^o, \cdot, 0)$ with respect to $GL(\mathbf{R}^m) \times GL(\mathbf{R}^n)$ and the invariant tensor theorem [2] we deduce that the value (2) is proportional to $Q(0, 0)$. That is why, $A_2(\Theta) = \tau Q$ for some $\tau \in \mathbf{R}$. □

From Proposition 2 it follows (immediately) the following

Proposition 3. *Any $\mathcal{FM}_{m,n}$-natural operator $A_1 : \tilde{J}^2 \rightsquigarrow J^1$ is of the form*

$$A_1(\Theta) = \Gamma + \tau Q$$

for some $\tau \in \mathbf{R}$, where $\Theta = (\Gamma, Q, G)$ is an arbitrary second order nonholonomic connection on $Y \to M$.

Then it remains to classify all $\mathcal{FM}_{m,n}$-natural operators $A_3 : \tilde{J}^2 \rightsquigarrow \otimes^2 T^*B \otimes V$ transforming second order nonholonomic connections $\Theta = (\Gamma, Q, G)$ on $Y \to M$ into tensor fields $A_3(\Gamma, Q, G) : Y \to \otimes^2 T^*M \otimes VY$.

Example 1. Let $\Theta = (\Gamma, Q, G)$ be a second order nonholonomic connection on $Y \to M$. We can take the curvature $C\Gamma = [\Gamma, \Gamma] : Y \to \wedge^2 T^*M \otimes VY$ of Γ, see Sect. 17.1 in [2]. The correspondence $D_1 : \tilde{J}^2 \rightsquigarrow \otimes^2 T^*B \otimes V$ given by $D_1(\Gamma, Q, G) = C\Gamma$ is a $\mathcal{FM}_{m,n}$-natural operator.

Example 2. The correspondence $D_2 : \tilde{J}^2 \rightsquigarrow \otimes^2 T^*B \otimes V$ given by

$$D_2(\Gamma, Q, G) = C(\Gamma + Q)$$

is a $\mathcal{FM}_{m,n}$-natural operator.

Example 3. We can take the alternation $Alt(G) : Y \to \wedge^2 T^*M \otimes VY$ of G. The correspondence $D_3 : \tilde{J}^2 \rightsquigarrow \otimes^2 T^*B \otimes V$ given by $D_3(\Gamma, Q, G) = Alt(G)$ is a $\mathcal{FM}_{m,n}$-natural operator.

Example 4. We can take the symmetrization $Sym(G) : Y \to S^2 T^*M \otimes VY$. The correspondence $D_4 : \tilde{J}^2 \rightsquigarrow \otimes^2 T^*B \otimes V$ given by $D_4(\Gamma, Q, G) = Sym(G)$ is a $\mathcal{FM}_{m,n}$-natural operator.

Proposition 4. *Any $\mathcal{FM}_{m,n}$-natural operator $A_3 : \tilde{J}^2 \rightsquigarrow \otimes^2 T^*B \otimes V$ is of the form*

$$A_3 = k_1 D_1 + k_2 D_2 + k_3 D_3 + k_4 D_4$$

for real numbers k_1, k_2, k_3, k_4.

Proof. Similarly as in the proof of Proposition 2, A_3 is uniquely determined by the values

$$A_3(\Gamma, Q, G)(0,0) \in \otimes^2 T_0^* \mathbf{R}^m \otimes V_{(0,0)}(\mathbf{R}^m \times \mathbf{R}^n) \tag{3}$$

for all first order connections Γ on $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ with $\Gamma(0,0) = \theta^o$, all tensor fields $Q : \mathbf{R}^m \times \mathbf{R}^n \to T^*\mathbf{R}^m \otimes V(\mathbf{R}^m \times \mathbf{R}^n)$ and all tensor fields $G : \mathbf{R}^m \times \mathbf{R}^n \to \otimes^2 T^*\mathbf{R}^m \otimes V(\mathbf{R}^m \times \mathbf{R}^n)$. Then using the non-linear Petree theorem [2] and the invariance of A_3 with respect to the homotheties $t id_{\mathbf{R}^m \times \mathbf{R}^n}$ for $t > 0$ and the homogeneous function theorem [2] and next the invariance of A_3 with respect to the fiber homotheties $id_{\mathbf{R}^m} \times t id_{\mathbf{R}^n}$ for $t > 0$ and the base homotheties $t id_{\mathbf{R}^m} \times id_{\mathbf{R}^n}$ for $t > 0$ we deduce that the values (3) are of the form

$$\begin{aligned} &A_3(\Gamma, 0, 0)(0,0) + A_3(\Gamma^o, \tilde{Q}, 0)(0,0) \\ &\qquad + A_3(\Gamma^o, 0, G^o)(0,0) + A_3(\Gamma^1, Q^o, 0)(0,0), \end{aligned} \tag{4}$$

where Γ^o is the trivial connection and Q^o is the constant tensor field such that $Q^o(0,0) = Q(0,0)$ and G^o is the constant tensor field such that $G^o(0,0) = G(0,0)$ and $\tilde{Q} = Q - Q^o$ and $\Gamma^1 = \Gamma^o + Q^1$ and $Q^1 : \mathbf{R}^m \times \mathbf{R}^n \to T^*\mathbf{R}^m \otimes V(\mathbf{R}^m \times \mathbf{R}^n)$ is some tensor field of the form

$$Q^1 = \sum_{k,l=1}^{n} \sum_{i=1}^{m} a_{i,l}^k y^l dx^i \otimes \frac{\partial}{\partial y^k}$$

with constant $a_{i,l}^k$ dependent on Γ. Moreover, the second summand of (4) depends on the first derivatives of $\tilde{Q}$ only, and the forth summand of (4) depends linearly on the $a^k_{i,l}$'s. In particular,

(5) $$A_3\Big(\Gamma^o, dx^i \otimes \frac{\partial}{\partial y^k}, 0\Big)(0,0) = 0$$

for all i, k as above, and the forth summand of (4) is determined by the values

(6) $$A_3\Big(\Gamma^o + y^l dx^i \otimes \frac{\partial}{\partial y^k}, Q^o, 0\Big)(0,0)$$

for all i, k, l and Q^o as above. The third summand of (4) (more explicitly, the map $G^o \to A_3(\Gamma^o, 0, G^o)$) can be treated as the $GL(m) \times GL(n)$-invariant map $\otimes^2(\mathbf{R}^m)^* \otimes \mathbf{R}^n \to \otimes^2(\mathbf{R}^m)^* \otimes \mathbf{R}^n$. Then (it is well known), it is a linear combination of the alternation and symmetrization. Similarly, the second summand of (4) can be also treated as the $GL(m) \times GL(n)$-invariant map $\otimes^2(\mathbf{R}^m)^* \otimes \mathbf{R}^n \to \otimes^2(\mathbf{R}^m)^* \otimes \mathbf{R}^n$. Then it is a linear combination of the alternation and symmetrization, too. But, using the invariance of A_3 with respect to $(x^1 + (x^1)^2, x^2, \dots, x^m, y^1, \dots, y^n)$, from (5) for $i = 1$ and $k = 1$ we obtain $A_3(\Gamma^o, x^1 dx^1, 0)(0,0) = 0$. Then the second summand of (4) corresponds only to a constant multiple of the alternation. Then replacing A_3 by $A_3 - k_2 D_2 - k_3 D_3 - k_4 D_4$ for some respective real numbers k_2, k_3, k_4 we may assume that the second and the third summands of (4) are zero. Then using the invariance of A_3 with respect to the $\mathcal{FM}_{m,n}$-map $(x^1, \dots, x^m, y^1, \dots, y^k + x^i y^l, \dots, y^n)$ (where only $m + k$-position is exceptional) from (5) (and the additional assumption that the second summand of (4) is zero) we deduce that the value (6) is zero for all i, k, l as above. Then the forth summand of (4) is zero, too. Then $A_3(\Gamma, Q, G)$ does not depend on G and Q. Then A_3 is determined by a $\mathcal{FM}_{m,n}$-natural operator $D : J^1 \rightsquigarrow \otimes^2 T^*B \otimes V$ given by $D(\Gamma) = A_3(\Gamma, 0, 0)$. But by Proposition 4 in [3], $D = k_1 C$ for some $k_1 \in \mathbf{R}$. Then $A_3 = k_1 D_1$. The proof is complete. □

Thus we have proved

Theorem 1. *Any $\mathcal{FM}_{m,n}$-natural operator $A : \tilde{J}^2 \rightsquigarrow \tilde{J}^2$ is of the form*

$$A(\Theta) = (\Gamma + \tau_1 Q, \tau_2 Q, k_1 C\Gamma + k_2 C(\Gamma + Q) + k_3 Alt(G) + k_4 Sym(G))$$

for some (uniquely determined by A) real numbers $\tau_1, \tau_2, k_1, k_2, k_3, k_4$, where $\Theta = (\Gamma, Q, G)$ is an arbitrary second order nonholonomic connection on $Y \to M$.

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