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RAYMOND MORTINI and RUDOLF RUPP

Sums of holomorphic selfmaps of the unit disk

Abstract. We derive for $p > 0$ the best constants c_p for which $\left|\frac{1+z}{2}\right| + c_p \left|\frac{1-z}{2}\right|^p \le 1$ whenever $|z| \le 1$. We also determine for $0 \le p \le 1$ all complex numbers c for which the functions $\frac{1+z}{2} + c\left(\frac{1-z}{2}\right)^p$ are selfmaps of the closed unit disk.

1. Introduction. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathbf{D} = \{z \in \mathbb{C} : |z| \le 1\}$ its closure. It is very easy to see that whenever $u(z) = (1+z)/2$ and $v(z) = (1-z)/2$, then $|u|^2 + |v|^2 \le 1$ on $\mathbf{D}$. These functions and its companions $u \circ p$, where p is a general peak-function in a uniform algebra, play an important role in studying isometric interpolation problems (see [2], [5]). But also in operator theory, combinations of powers of u and v were chosen to study algebraic and functional analytic properties of composition operators on various spaces of analytic functions (see [4], [1] and [3]). In [4, p. 492], a paper that served as the impetus for our study of the class of functions $u + cv^p$, the authors assert that for every $p > 0$ the function $(1+z)/2 + c[(z-1)/2]^p$ is a selfmap for $\mathbb{D}$ whenever $c > 0$ is small. We will show, among other things, that for $0 \le p < 1$, the maps $u + cv^p$ are selfmaps of the unit disk if and only if c belongs to a certain convex subset R_p of the disk $|z + 1/2| \le 1/2$.

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2. The functions $|u|+c|v|^p$. Let $u(z)=(1+z)/2$ and $v(z)=(1-z)/2$. In this section we will study the sum $|u|+c|v|^p$ for $c>0$ and $p>0$, considered as a function on $\mathbf{D}$.

Proposition 2.1. *The following assertions are true:*

(i) $|u|+\frac{1}{2}|v|^p \le 1$ *on* $\mathbf{D}$ *if* $p\ge 2$.
(ii) $\max_{\mathbf{D}}[|u|+c|v|^p]>1$ *for every* p *with* $0<p<2$ *and every* $c>0$.
(iii) *The best possible constant* $c>0$ *for which* $|u|+c|v|^p\le 1$ *in* $\mathbf{D}$ *is*

$$c_p := \frac{(p-1)^{p-1}}{p^{p/2}\,(p-2)^{(p-2)/2}}$$

whenever $p>2$ *and* $c=1/2$ *whenever* $p=2$.

Proof. (i) Due to the maximum principle for subharmonic functions, and symmetry, it is sufficient to evaluate $\Delta(z)=|u(z)|+c|v(z)|^p$ at $z=e^{i\theta}$ where $0\le\theta\le\pi$. Note that $\Delta(z)=\cos(\theta/2)+c\sin^p(\theta/2)$. Now for fixed $p\ge 2$ and $c\in\,]0,\frac{1}{2}]$ we have

$$\begin{aligned}\Delta(z) &\le \cos\frac{\theta}{2}+\frac{1}{2}\sin^2\frac{\theta}{2} = -\frac{1}{2}\cos^2\frac{\theta}{2}+\cos\frac{\theta}{2}+\frac{1}{2}\\ &= 1-\frac{1}{2}\left(\cos\frac{\theta}{2}-1\right)^2 \le 1\end{aligned}$$

on the interval $[0,\pi]$.

(ii) Now let $0<p<2$. We put $y=\sin(\theta/2)$, $0\le y\le 1$. Then

$$\begin{aligned}&\Delta(e^{i\theta})=\sqrt{1-y^2}+cy^p\le 1\\ &\iff 1-y^2\le 1+c^2y^{2p}-2cy^p\\ &\iff cy^{p-2}(2-cy^p)\le 1.\end{aligned}$$

Noticing that $1\le 2-cy^p\le 2$, we see that for all c, $0<c\le 1$, there exists y (close to 0), such that $cy^{p-2}(2-cy^p)>1$. This gives (ii).

(iii) We are looking for the largest $c:=c_p$ such that $\cos\frac{\theta}{2}+c\left(\sin\frac{\theta}{2}\right)^p\le 1$ on $[0,\pi]$; that is,

$$c\le\frac{1-\cos\frac{\theta}{2}}{(\sin\frac{\theta}{2})^p}=2^{1-p}\cdot\sqrt{\frac{(\sin^2\frac{\theta}{4})^{2-p}}{(1-\sin^2\frac{\theta}{4})^p}},\quad 0<\theta\le\pi.$$

Let $H(t)=\frac{t^{2-p}}{(1-t)^p}$. If $p=2$, then $\min_{0<t\le\frac{1}{2}}H(t)=1$; hence $c_2=\frac{1}{2}$. If $p>2$, then $t_p:=\frac{p-2}{2(p-1)}<\frac{1}{2}$, and

$$\min_{0<t\le\frac{1}{2}}H(t)=\frac{1}{\max_{0<t\le\frac{1}{2}}t^{p-2}\,(1-t)^p}=\frac{1}{t_p^{p-2}\,(1-t_p)^p}.$$

It follows that for $p > 2$

$$c_p = 2^{1-p} \cdot \sqrt{\frac{2^{2p-2}(p-1)^{2p-2}}{p^p(p-2)^{p-2}}} = \frac{(p-1)^{p-1}}{p^{p/2}(p-2)^{(p-2)/2}}. \qquad \square$$

Remark. We note that $\lim_{p\to 2} c_p = 1/2$, that c_p is increasing in p, and that $\lim_{p\to\infty} c_p = 1$.

3. The functions $u + cv^p$, $p > 0$, $c > 0$. For $p > 0$ and $c > 0$ let $f_{p,c} = u + cv^p$, where we choose the branch of the logarithm of w, $\operatorname{Re} w > 0$, that satisfies $\log 1 = 0$ in order to define v^p (note that $\operatorname{Re} v > 0$ in $\mathbf{D} \setminus \{1\}$). We are interested in the problem of when $f_{p,c}$ is a selfmap of $\mathbf{D}$. For example, if $c > 1$, then $f_{p,c}(-1) = c > 1$, so $f_{p,c}$ is not a selfmap of $\mathbf{D}$. Thus we may assume throughout this section that $0 < c \le 1$.

Proposition 3.1. *The following assertions are true:*

(i) *$f_{p,c}$ is not a selfmap of $\mathbf{D}$ if $0 < p < 1$ and $0 < c \le 1$.*
(ii) *$f_{p,c}$ is a selfmap of $\mathbf{D}$ for every $1 \le p \le 3$ and every $0 < c \le 1$.*

Proof. (i) Let $0 < p < 1$. Then for $0 < x < 1$ we have

$$\frac{1+x}{2} + c\left(\frac{1-x}{2}\right)^p \le 1 \Longleftrightarrow 2^{1-p}c \le (1-x)^{1-p},$$

which is not satisfied for x close to 1.

(ii) If $p = 1$, then for $0 < c \le 1$,

$$\left|\frac{1+z}{2} + c\frac{1-z}{2}\right| = \left|\frac{1}{2} + \frac{c}{2} + z\left(\frac{1}{2} - \frac{c}{2}\right)\right| \le \frac{1}{2} + \frac{c}{2} + \frac{1}{2} - \frac{c}{2} = 1.$$

Let $1 < p \le 3$ and $c = 1$. As above, we need only consider the case where $z = e^{i\theta}$ with $0 < \theta < \pi$. Then

$$|f_{p,1}(e^{i\theta})| = \left|\cos\frac{\theta}{2} - i\sin^p\frac{\theta}{2}\, e^{i(p-1)(\theta-\pi)/2}\right|.$$

Hence

$$\left|f_{p,1}(e^{i\theta})\right|^2 = \cos^2\frac{\theta}{2} + \sin^{2p}\frac{\theta}{2} + 2\cos\frac{\theta}{2}\sin^p\frac{\theta}{2}\sin\varphi,$$

where $\varphi = (p-1)(\theta-\pi)/2$. Since $1 < p \le 3$, we have that $-\pi \le \varphi \le 0$. Hence $\sin\varphi \le 0$. Thus

$$|f_{p,1}(e^{i\theta})|^2 \le \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1.$$

Now let $1 < p \le 3$ and $0 < c < 1$. We fix two points u and w in $\mathbb{D}$ (for instance $u = u(z)$ and $w = v^p(z)$ for some $z \in \mathbb{D}$.) Note that the case $c = 1$ above implies that $u + w \in \mathbb{D}$. Since $\mathbb{D}$ is convex, the line segment joining $u \in \mathbb{D}$ and $u + w \in \mathbb{D}$, given by $\{u + cw : 0 \le c \le 1\}$, is contained in $\mathbb{D}$. Thus the function $f_{p,c}$ is a selfmap of $\mathbb{D}$. $\square$

We remark that $c = 1/2$ is the best constant in

$$\left|\frac{1+z}{2}\right| + c\left|\frac{1-z}{2}\right|^2 \le 1,$$

but that c can be chosen to be 1 in

$$\left|\frac{1+z}{2} + c\left(\frac{1-z}{2}\right)^2\right| \le 1.$$

(Note that $\left|\frac{1+z}{2} + \left(\frac{1-z}{2}\right)^2\right| = \left|\frac{3}{4} + \frac{z^2}{4}\right| \le 1$.)

We guess that for all $p \ge 1$ we have

$$\left|\frac{1+z}{2} + \left(\frac{1-z}{2}\right)^p\right| \le 1.$$

In addition to the case $1 \le p \le 3$ we considered above, we can also confirm this inequality for $p = 4$ and $p = 5$.

4. The functions $u + cv^p$, $0 \le p \le 1$, $c \in \mathbb{C}$. In this section we determine all complex numbers c for which $f_{p,c}$ is a selfmap of $\mathbf{D}$, whenever $0 \le p \le 1$.

Lemma 4.1. *Let $0 \le p < 1$. Then the regions*

$$R_p := \left\{\left(\frac{1-a}{2}\right)^{1-p} : a \in \mathbf{D}\right\}$$

are strictly decreasing. For $p = 0$, the set R_0 coincides with the closed disk centered at $z = 1/2$ and radius $1/2$.

Proof. First we note that the function M_p defined by $M_p(a) = \left(\frac{1-a}{2}\right)^{1-p}$ is a conformal map of $\mathbb{D}$ onto the interior of R_p. If $p = 0$, then R_p is the disk $\{z \in \mathbf{D} : |z - 1/2| \le 1/2\}$. The boundary of R_0 can be represented in polar coordinates by $z(t) = e^{it}\cos t$, $-\pi/2 \le t \le \pi/2$. Consider the principal branch of the logarithm. Then $L := \log M_0(\mathbb{D})$ is an unbounded convex domain in the left half-plane, contained in the strip $\{w \in \mathbb{C} : |\text{Im } w| < \frac{\pi}{2}\}$. The upper half of the boundary of L is given by the curve $\mathfrak{C}^+$, parametrised as $\log\cos t + it$, $0 \le t \le \pi/2$. The lower half $\mathfrak{C}^-$ of the boundary is the reflection of $\mathfrak{C}^+$ with respect to the real axis. The horizontal asymptotes are the lines $\text{Im } w = \pm\frac{\pi}{2}$. Due to convexity, and the fact that $0 \in L$, the image $\log M_p(\mathbf{D}) = (1-p)\log M_0(\mathbf{D})$ is contained in $\log M_0(\mathbf{D})$ (see figure). Hence $R_p \subseteq R_0$. The same reasoning works for the pairs (p, p'), $0 \le p < p' < 1$, instead of $(0, p)$. Hence $R_{p'} \subseteq R_p$. $\square$

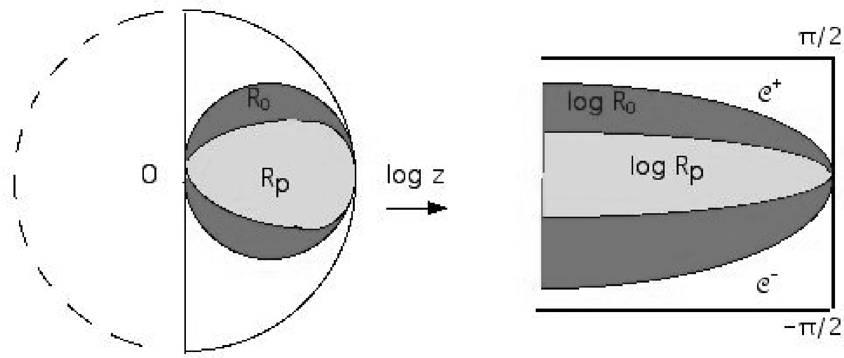

FIGURE 1. The regions $\log R_p$ and $\log R_0$.

Lemma 4.2. *Let $\mathfrak{C}$ be the boundary of the domain $-R_p$, $0 \le p < 1$. Then $w_p(t) := e^{it} \sin^{1-p}\left(\frac{t-\frac{\pi}{2}(p+1)}{1-p}\right)$, $\frac{\pi}{2}(1+p) \le t \le \pi$, is the polar representation of the upper half of $\mathfrak{C}$.*

Proof. We assume that $0 \le \theta \le \pi$. Then

$$M_p(e^{i\theta}) = \left(e^{i(\frac{\theta-\pi}{2})} \sin\frac{\theta}{2}\right)^{1-p}.$$

Hence $|M_p(e^{i\theta})| = \sin^{1-p}(\theta/2)$ and $\arg(-M_p(e^{i\theta})) = \pi - \frac{\pi-\theta}{2}(1-p)$.

Let $t = \arg(-M_p(e^{i\theta}))$. Then $t \in [\frac{\pi}{2}(1+p), \pi]$ and

$$\theta = \frac{2t - \pi(p+1)}{1-p}.$$

Thus

$$w_p(t) := -M_p(e^{i\theta}) = e^{it} \sin^{1-p}\left(\frac{t - \frac{\pi}{2}(p+1)}{1-p}\right).$$

If $p = 0$, we get $w_0(t) = e^{it}\sin(t - \frac{\pi}{2}), \frac{\pi}{2} \le t \le \pi$. It is easy to see that $-R_0$ is the disk $|z + 1/2| \le 1/2$. □

Theorem 4.3. i) *Let $0 < p < 1$ and $c \in \mathbb{C}$. Then the function*

$$f_{p,c}(z) = \frac{1+z}{2} + c\left(\frac{1-z}{2}\right)^p$$

is a selfmap of $\mathbf{D}$ if and only if $c \in -R_p$; that is, if $c = -\left(\frac{1-a}{2}\right)^{1-p}$ for some $a \in \mathbf{D}$. In particular, if $|c| = 1$, then $f_{p,c}$ is a selfmap of $\mathbf{D}$ if and only if $c = -1$.

ii) *For $p = 0$, $(1+z)/2 + c$ is a selfmap of $\mathbf{D}$ if and only if $|c+1/2| \le 1/2$.*

iii) *For $p = 1$, $(1+z)/2 + c(1-z)/2$ is a selfmap of $\mathbf{D}$ if and only if $-1 \le c \le 1$.*

Proof. First we show that whenever $f_{p,c}$ is a selfmap of $\mathbf{D}$ and $0 \le p < 1$, then $c = -\left(\frac{1-a}{2}\right)^{1-p}$ for some $a \in \mathbf{D}$. To this end we use the Denjoy–Wolff theorem: If $f_{p,c}$ is a selfmap of $\mathbf{D}$ that is not the identity $f_{1,-1}$, then either it has a unique fixed point in $\mathbb{D}$ or it has a unique boundary fixed point b with the property that the angular derivative at b is strictly positive and less than or equal to 1. Now our $f_{p,c}$ always has 1 as a fixed point; however the angular derivative does not exist at that point. Thus we must look for other fixed points of $f_{p,c}$ in $\mathbf{D}$.

So let $f_{p,c}(a) = a$. Then $a = 1$ or $1 - a + \frac{2c}{2^p}(1-a)^p = 0$. The latter is equivalent to

$$c = -\left(\frac{1-a}{2}\right)^{1-p}. \tag{4.1}$$

Thus, a necessary condition for $f_{p,c}$ being a selfmap of $\mathbf{D}$, is that c belongs to the region

$$R_p^* := \left\{ -\left(\frac{1-a}{2}\right)^{1-p} : a \in \mathbf{D} \right\}.$$

Note that if $|c| = 1$, then (4.1) implies that $a = c = -1$. To deal with the case $p = 1$, we proceed in another way. To begin with, let p be arbitrary, $0 < p \le 1$.

First we note that $\overline{\left(\frac{1-z}{2}\right)^p} = \left(\frac{1-\bar{z}}{2}\right)^p$. Hence it suffices to deal with those parameters c that belong to the closed upper half plane. Moreover, since $f_{p,c}(-1) = c$, we can restrict to parameters c that are in the closed unit disk. Let $c = re^{i\varphi}$, where $0 \le \varphi \le \pi$, $0 < r \le 1$.

If $c = 0$, there is nothing to show. So suppose $c \ne 0$. For $z = e^{i\theta}$, $0 \le \theta \le \pi$, we have:

$$f_{p,c}(e^{i\theta}) = e^{i\theta/2} \cos\frac{\theta}{2} + re^{i\varphi} \sin^p \frac{\theta}{2}\, e^{ip(\theta-\pi)/2}.$$

Hence

$$\begin{aligned}
|f_{p,c}(e^{i\theta})|^2 &= \cos^2\frac{\theta}{2} + r^2 \sin^{2p}\frac{\theta}{2} \\
&\qquad + 2r\cos\frac{\theta}{2}\sin^p\frac{\theta}{2}\cos\left[\frac{p-1}{2}(\theta-\pi) - \frac{\pi}{2} + \varphi\right] \\
&= \cos^2\frac{\theta}{2} + r^2 \sin^{2p}\frac{\theta}{2} \\
&\qquad + 2r\cos\frac{\theta}{2}\sin^p\frac{\theta}{2}\sin\left[\frac{p-1}{2}(\theta-\pi) + \varphi\right].
\end{aligned}$$

Now

$$|f_{p,c}(e^{i\theta})|^2 \le 1 \iff r^2\sin^{2p}\frac{\theta}{2} + 2r\cos\frac{\theta}{2}\sin^p\frac{\theta}{2}\sin\left[\frac{p-1}{2}(\theta-\pi)+\varphi\right] \le \sin^2\frac{\theta}{2}.$$

For $\theta \neq 0$ we divide by $\sin^p(\theta/2)$, which yields

$$r^2 \sin^p \frac{\theta}{2} + 2r \cos \frac{\theta}{2} \sin \left[\frac{p-1}{2}(\theta - \pi) + \varphi\right] \leq \sin^{2-p} \frac{\theta}{2}.$$

Letting $\theta \to 0^+$, gives

$$2r \sin\left(\frac{1-p}{2}\pi + \varphi\right) \leq 0. \tag{4.2}$$

Thus $\frac{p+1}{2}\pi \leq \varphi \leq \frac{p+3}{2}\pi$. Hence, if $p = 1$ we may use our hypothesis that $0 \leq \varphi \leq \pi$, to see that $\varphi = 0$ or $\varphi = \pi$. Thus $c \in [-1, 1]$.

Next we prove the sufficiency of these conditions.

• Let $p = 0$ and $|c + 1/2| \leq 1/2$. Then $c = -1/2 + (1/2)r\xi$, where $0 \leq r \leq 1$ and $|\xi| = 1$. Hence

$$\left|\frac{1+z}{2} + c\right| = \left|\frac{z}{2} + \frac{r}{2}\xi\right| \leq \frac{|z| + |\xi|}{2} \leq 1.$$

• If $p = 1$, and $-1 \leq c \leq 1$, then trivially

$$\left|\frac{1+z}{2} + c\frac{1-z}{2}\right| = \left|\frac{1+c}{2} + z\frac{1-c}{2}\right| \leq \frac{1+c}{2} + \frac{1-c}{2} \leq 1.$$

• Now let $0 < p < 1$ and suppose that c is located in the closed region $R_p^* := \left\{-\left(\frac{1-a}{2}\right)^{1-p} : a \in \mathbf{D}\right\}$. Let $A = \frac{1-a}{2}$, $B = \frac{1-z}{2}$ for $a, z \in \mathbf{D}$. We show that $C := A^{1-p}B^p$ belongs to the disk $\Delta = \{|z - 1/2| \leq 1/2\}$. First note that $\operatorname{Re} C \geq 0$. If log denotes the principal branch of the logarithm on the right-half plane, we obtain that $\log(A^{1-p}B^p) = (1-p)\log A + p \log B$. Since the domain $L = \log \Delta$ in Lemma 4.1 above is convex, we get that $(1-p)\log A + p\log B \in L$. Hence $A^{1-p}B^p \in \Delta$. Thus, by the case $p = 0$, we conclude that

$$\frac{1+z}{2} - \left(\frac{1-a}{2}\right)^{1-p}\left(\frac{1-z}{2}\right)^p = \frac{1+z}{2} - C \in \mathbf{D}. \qquad \square$$

The previous result shows that a statement in MacCluer, Ohno and Zhao [4] is not correct:

The function $f_{1/2,ir} = \frac{1+z}{2} + ri\sqrt{\frac{1-z}{2}}$ (principal branch) is not a selfmap of $\mathbf{D}$, however small $r > 0$ is. In particular, there exists $z \in \mathbf{D}$ such that $\overline{f_{1/2,i}(z)} = \frac{1+\bar{z}}{2} - i\sqrt{\frac{1-\bar{z}}{2}} \notin \mathbf{D}$. Thus the function $f_{1/2,-i}$ is not a selfmap, either.

More generally, let $p \in\ ]0,1[$. Then none of the maps $\frac{1+z}{2} + t\,(z-1)^p$ considered in [4] is a selfmap of $\mathbb{D}$ whenever $t > 0$. In fact, for $t > 0$, write the function $\frac{1+z}{2} + t\,(z-1)^p$ as $\frac{1+z}{2} + re^{i\pi p}\left(\frac{1-z}{2}\right)^p$. Then the parameter $c = re^{i\pi p}$ does not lie in the domain R_p^*, since its argument is πp and $\pi p < \frac{\pi}{2}(1+p)$. Our statement now follows from Theorem 4.3.

5. Convex perturbations. In [4] functions of the type $sz + 1 - s$ for $0 < s < 1$ are considered, too. Here we have the following result. Recall that the disk algebra $A(\mathbb{D})$ is the space of all functions continuous on $\mathbf{D}$ and holomorphic in $\mathbb{D}$.

Proposition 5.1. *Let $f \in A(\mathbb{D})$ be a function such that $(1+z)/2 + f(z)$ is a selfmap of $\mathbf{D}$. Then, for every $s \in]0,1[$ there exists a constant $c_s > 0$ such that $(sz + 1 - s) + cf$ is a selfmap of $\mathbf{D}$ for every c with $0 \leq c \leq c_s$.*

Proof. Let $(\alpha, \beta) \in]0,1[^2$ satisfy $\alpha + \beta = 1$. Since $\mathbf{D}$ is convex, we have that for every $\sigma \in [0,1]$ and $z \in \mathbf{D}$

$$h(z) = \alpha(\sigma z + 1 - \sigma) + \beta \left[\frac{1+z}{2} + f(z)\right] \in \mathbf{D}.$$

But

$$h(z) = \left(\alpha\sigma + \frac{\beta}{2}\right) z + \left(\alpha - \alpha\sigma + \frac{\beta}{2}\right) + \beta f(z).$$

Now we have to choose α, β and σ such that $s = \alpha\sigma + \beta/2$. Then, automatically, $1 - s = \alpha - \alpha\sigma + \beta/2$. To see that such a choice is possible, we use the assumption that $\beta = 1 - \alpha$ to obtain that $\alpha = (s - 1/2)/(\sigma - 1/2)$. If we now choose σ so that, either $1/2 < s < \sigma \leq 1$, or $0 \leq \sigma < s < 1/2$, then $0 < \alpha < 1$. Now we may define c_s by $c_s := \beta = 1 - \alpha$ to conclude that $(sz + 1 - s) + c_s f$ is a selfmap of $\mathbf{D}$. For example, if $s > 1/2$ and $\sigma = 1$, then $c_s = 2(1 - s)$; if $0 < s < 1/2$ and $\sigma = 0$, then $c_s = 2s$.

Once we have found a constant c_s for which $(sz + 1 - s) + c_s f$ is a selfmap of $\mathbf{D}$, it is now easy to see that for every c with $0 \leq c \leq c_s$, the map $(sz + 1 - s) + cf$ is a selfmap, too. In fact, since $\mathbf{D}$ is starlike with respect to any point $a \in \mathbf{D}$, it follows that $a + tb \in \mathbf{D}$ whenever $a + t_0 b \in \mathbf{D}$ and $0 \leq t \leq t_0$. Hence we get Proposition 5.1. □

We mention here that whenever $u + cv^p$ and $u + c'v^p$ are selfmaps of $\mathbf{D}$, then for any convex combination $c'' := sc + (1 - s)c'$ of the points $c, c' \in \mathbf{D}$, $0 < s < 1$, we have that $u + c''v^p$ is a selfmap of $\mathbf{D}$, too. In fact, $u + c''v^p = s(u + cv^p) + (1 - s)(u + c'v^p)$ is a convex combination of such functions. As an application we mention that by Propositions 2.1 and 3.1, $\frac{1+z}{2} + c\left(\frac{1-z}{2}\right)^2$ is a selfmap of $\mathbf{D}$ if c belongs to the convex hull of the disk $\{|z| \leq 1/2\}$ and the point 1. Is this also a necessary condition?

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Raymond Mortini
Département de Mathématiques
Université Paul Verlaine
Ile du Saulcy
F-57045 Metz, France
e-mail: `mortini@math.univ-metz.fr`

Rudolf Rupp
Fachbereich Allgemeinwissenschaften
Georg-Simon-Ohm-Hochschule Nürnberg
Kesslerplatz 12, D-90489 Nürnberg
Germany
e-mail: `rudolf.rupp@ohm-hochschule.de`

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