ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXI, 2007 SECTIO A 117–127

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Differential sandwich theorems for analytic functions defined by Hadamard product

Abstract. In the present investigation, we obtain some subordination and superordination results involving Hadamard product for certain normalized analytic functions in the open unit disk. Our results extend corresponding previously known results.

1. Introduction. Let $\mathcal{H}$ be the class of analytic functions in $\Delta := \{z : |z| < 1\}$ and $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_2 z^2 + \dots$. Let $p, h \in \mathcal{H}$ and let $\phi(r,s,t;z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1.1}$$

then p is a solution of the differential superordination (1.1). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [14] obtained conditions on h, q

2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C80.

Key words and phrases. Univalent functions, starlike functions, convex functions, differential subordination, differential superordination, Hadamard product (convolution), Dziok–Srivastava linear operator.

and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z).$$

For $\alpha_j \in \mathbb{C} \quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots, m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$$ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1;\ l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0; \\ a(a+1)(a+2)\ldots(a+n-1), & n \in \mathbb{N} := \{1, 2, 3 \ldots\}. \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok–Srivastava operator [6] (see also [7, 24]) $H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by the Hadamard product

$$\begin{aligned} H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) &:= h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!}. \end{aligned} \tag{1.2}$$

For brevity, we write

$$H_m^l[\alpha_1] f(z) := H_m^l(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z).$$

It is easy to verify from (1.2) that

$$z(H_m^l[\alpha_1] f(z))' = \alpha_1 H_m^l[\alpha_1 + 1] f(z) - (\alpha_1 - 1) H_m^l[\alpha_1] f(z). \tag{1.3}$$

Special cases of the Dziok–Srivastava linear operator includes the Hohlov linear operator [8], the Carlson–Shaffer linear operator $L(a, c)$ [5], the Ruscheweyh derivative operator D^n [22], the generalized Bernardi–Libera–Livingston linear integral operator (cf. [2], [11], [12]) and the Srivastava–Owa fractional derivative operators (cf. [17], [18]).

Lewandowski et al. [9], Li and Owa [10], Nunokawa et al. [16], Padamanbhan [19], Ramesha et al. [20] and Ravichandran et al. [21] have found out sufficient conditions for functions to be starlike. Further, using the results of Miller and Mocanu [14], Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators (see [3]). Recently many authors [1, 15, 23] have used the results of Bulboacă [4] and shown some sufficient conditions applying first order differential subordinations and superordinations.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions $f(z)$ in Δ such that $(f * \Psi)(z) \neq 0$ and f to satisfy

$$q_1(z) \prec \frac{H^l_m[\alpha_1 + 1](f * \Phi)(z)}{H^l_m[\alpha_1](f * \Psi)(z)} \prec q_2(z),$$

where q_1, q_2 are given univalent functions in Δ and $\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$, $\Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ are analytic functions in Δ with $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$. Also, we obtain the number of known results as their special cases.

2. Subordination results. For our present investigation, we shall need the following:

Definition 2.1 ([14])**.** Denote by Q, the set of all functions f that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\Delta : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 2.2 ([13])**.** *Let q be univalent in the unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set*

$$\psi(z) := zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) := \theta(q(z)) + \psi(z).$$

Suppose that

(1) *$\psi(z)$ is starlike univalent in Δ and*
(2) $\mathrm{Re}\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ *for $z \in \Delta$.*

If p is analytic with $p(0) = q(0)$, $p(\Delta) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Lemma 2.3 ([4])**.** *Let q be convex univalent in the unit disk Δ and ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

(1) $\mathrm{Re}\{\vartheta'(q(z))/\varphi(q(z))\} > 0$ *for $z \in \Delta$ and*
(2) *$\psi(z) = zq'(z)\varphi(q(z))$ is starlike univalent in Δ.*

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$, with $p(\Delta) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.2}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Using Lemma 2.2, we first prove the following theorem.

Theorem 2.4. *Let $\Phi, \Psi \in \mathcal{A}$, $\alpha \neq 0$ and $\beta > 0$ be the complex numbers and $q(z)$ be convex univalent in Δ with $q(0) = 1$. Further assume that*

$$\mathrm{Re}\left\{\frac{\beta-\alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in \Delta). \tag{2.3}$$

If $f \in \mathcal{A}$ satisfies

$$\Upsilon(f,\, \Phi, \Psi,\, \alpha,\, \beta) \prec \alpha q^2(z) + (\beta-\alpha)q(z) + \alpha z q'(z), \tag{2.4}$$

where

$$\Upsilon(f,\Phi,\Psi,\alpha,\beta) := \begin{cases} (\beta-2\alpha)\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} + \alpha\left(\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}\right)^2 \\ +\alpha(\alpha_1+1)\frac{H_m^l[\alpha_1+2](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \\ -\alpha\alpha_1\frac{H_m^l[\alpha_1+1](f*\Psi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}\left(\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}\right), \end{cases} \tag{2.5}$$

then

$$\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \quad (z \in \Delta). \tag{2.6}$$

Then the function $p(z)$ is analytic in Δ and $p(0) = 1$. Therefore, by making use of (2.6), we obtain

$$\begin{aligned} &\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}\Bigg[\beta - 2\alpha + \alpha\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \\ &\quad + \alpha(\alpha_1+1)\frac{H_m^l[\alpha_1+2](f*\Phi)(z)}{H_m^l[\alpha_1+1](f*\Phi)(z)} - \alpha\alpha_1\frac{H_m^l[\alpha_1+1](f*\Psi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}\Bigg] \\ &\quad = \alpha p^2(z) + (\beta-\alpha)p(z) + \alpha z p'(z). \end{aligned} \tag{2.7}$$

By using (2.7) in (2.4), we have

$$\alpha p^2(z) + (\beta-\alpha)p(z) + \alpha z p'(z) \prec \alpha q^2(z) + (\beta-\alpha)q(z) + \alpha z q'(z). \tag{2.8}$$

By setting

$$\theta(w) := \alpha\omega^2 + (\beta-\alpha)\omega \quad \text{and} \quad \phi(\omega) := \alpha,$$

it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}-\{0\}$ and that $\phi(w) \neq 0$. Hence the result now follows by an application of Lemma 2.2. $\square$

When $l = 2$, $m = 1$, $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$ in Theorem 2.4, we state the following corollary.

Corollary 2.5. *Let* $\alpha \neq 0$, $\beta > 0$ *and* q *be convex univalent in* Δ *with* $q(0) = 1$ *and* (2.3) *holds true. If* $f \in \mathcal{A}$ *satisfies*

$$\Upsilon_1(f,\, \Phi, \Psi,\, \alpha,\, \beta) \prec \alpha q^2(z) + (\beta - \alpha)q(z) + \alpha z q'(z)$$

where

$$\Upsilon_1(f,\Phi,\Psi,\alpha,\beta) := \begin{cases} (\beta-2\alpha)\frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)} + \alpha\left(\frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)}\right)^2 \\ +\alpha(a+1)\frac{L(a+2,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)} \\ -a\alpha\frac{L(a+1,c)(f*\Psi)(z)}{L(a,c)(f*\Psi)(z)}\left(\frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)}\right), \end{cases} \tag{2.9}$$

then

$$\frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)} \prec q(z)$$

and q *is the best dominant.*

By fixing $\Phi(z) = \frac{z}{1-z}$ and $\Psi(z) = \frac{z}{1-z}$ in Theorem 2.4, we obtain the following corollary.

Corollary 2.6. *Let* $\alpha \neq 0$, $\beta > 0$ *and* q *be convex univalent in* Δ *with* $q(0) = 1$ *and* (2.3) *holds true. If* $f \in \mathcal{A}$ *satisfies*

$$\begin{aligned} \alpha(1-\alpha_1)\left(\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)}\right)^2 &+ (\beta-2\alpha)\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} \\ &+ \alpha(\alpha_1+1)\frac{H_m^l[\alpha_1+2]f(z)}{H_m^l[\alpha_1]f(z)} \\ &\prec \alpha q^2(z) + (\beta-\alpha)q(z) + \alpha z q'(z), \end{aligned}$$

then

$$\frac{H_m^l[\alpha_1+1]f(z)}{H_m^l[\alpha_1]f(z)} \prec q(z)$$

and q *is the best dominant.*

By taking $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 2.4, we state the following corollary.

Corollary 2.7. *Let $\alpha \neq 0$, $\beta > 0$ and q be convex univalent in Δ with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)}\left[\beta + \alpha\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)} + \alpha\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)} - \alpha\frac{z(f*\Psi)'(z)}{(f*\Psi)(z)}\right] \prec \alpha q^2(z) + (\beta-\alpha)q(z) + \alpha z q'(z),$$

then

$$\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \Psi(z)$ in Corollary 2.7, we obtain the following corollary.

Corollary 2.8. *Let $\alpha \neq 0$, $\beta > 0$ and q be convex univalent in Δ with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\beta\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} + \alpha\frac{z^2(f*\Phi)''(z)}{(f*\Phi)(z)} \prec \alpha q^2(z) + (\beta-\alpha)q(z) + \alpha z q'(z),$$

then

$$\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \frac{z}{1-z}$ in Corollary 2.8, we obtain the following corollary.

Corollary 2.9. *Let $\alpha \neq 0$, $\beta > 0$ and q be convex univalent in Δ with $q(0) = 1$ and (2.3) holds true. If $f \in \mathcal{A}$ satisfies*

$$\beta\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \alpha q^2(z) + (\beta-\alpha)q(z) + \alpha z q'(z),$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

By taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Corollary 2.9, we have the following corollary.

Corollary 2.10. *Let $0 < \alpha \leq \beta$ and $-1 \leq B < A \leq 1$. If*

$$\beta\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \frac{\beta + [A(\beta+2\alpha) + B(\beta-2\alpha)]z + A[(\beta-\alpha)B + \alpha A]z^2}{(1+Bz)^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}\ .$$

Remark 2.11. For the choices of $\beta = 1$, $A = 1$ and $B = -1$, Corollary 2.10 leads to the sufficient condition for starlike functions obtained in [19].

3. Superordination and sandwich results. Now, by applying Lemma 2.3, we prove the following theorem.

Theorem 3.1. *Let $\alpha \neq 0$ and $\beta > 0$. Let q be convex univalent in Δ with $q(0) = 1$. Assume that*

$$\mathrm{Re}\,\{q(z)\} \geq \mathrm{Re}\left\{\frac{\alpha-\beta}{2\alpha}\right\}. \tag{3.1}$$

Let $f \in \mathcal{A}$, $\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \in H[q(0),1] \cap Q$. Let $\Upsilon(f,\Phi,\Psi,\alpha,\beta)$ be univalent in Δ and

$$(\beta-\alpha)q(z) + \alpha q^2(z) + \alpha z q'(z) \prec \Upsilon(f,\Phi,\Psi,\alpha,\beta), \tag{3.2}$$

where $\Upsilon(f,\Phi,\Psi,\alpha,\beta)$ is given by (2.5), then

$$q(z) \prec \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}$$

and q is the best subordinant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)}. \tag{3.3}$$

Simple computation from (3.3), we get,

$$\Upsilon(f,\Phi,\Psi,\alpha,\beta) = (\beta-\alpha)p(z) + \alpha p^2(z) + \alpha z p'(z),$$

then

$$(\beta-\alpha)q(z) + \alpha q^2(z) + \alpha z q'(z) \prec (\beta-\alpha)p(z) + \alpha p^2(z) + \alpha z p'(z).$$

By setting $\vartheta(w) = \alpha w^2 + (\beta-\alpha)w$ and $\phi(w) = \alpha$, it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$. Also, $\phi(w)$ is analytic in $\mathbb{C} - \{0\}$ and that $\phi(w) \neq 0$.

Since $q(z)$ is convex univalent function, it follows that

$$\mathrm{Re}\left\{\frac{\vartheta'(q(z))}{\phi(q(z))}\right\} = \mathrm{Re}\left\{\frac{\beta-\alpha}{\alpha} + 2q(z)\right\} > 0, \quad z \in \Delta.$$

Now Theorem 3.1 follows by applying Lemma 2.3. □

When $l = 2$, $m = 1$, $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$ in Theorem 3.1, we state the following corollary.

Corollary 3.2. *Let $\alpha \neq 0$ and $\beta \geq 1$. Let q be convex univalent in Δ with $q(0) = 1$ and (3.1) holds true. If $f \in \mathcal{A}$ and*

$$(\beta-\alpha)q(z) + \alpha q^2(z) + \alpha z q'(z) \prec \Upsilon_1(f,\Phi,\Psi,\alpha,\beta),$$

where $\Upsilon_1(f,\Phi,\Psi,\alpha,\beta)$ *is given by* (2.9)*, then*

$$q(z) \prec \frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)}$$

and q *is the best subordinant.*

When $l=2$, $m=1$, $\alpha_1=1$, $\alpha_2=1$ and $\beta_1=1$ in Theorem 3.1, we derive the following corollary.

Corollary 3.3. *Let* $\alpha \neq 0$ *and* $\beta \geq 1$*. Let* q *be convex univalent in* Δ *with* $q(0)=1$ *and* (3.1) *holds true. If* $f \in \mathcal{A}$ *and*

$$\begin{aligned}&(\beta-\alpha)q(z)+\alpha q^2(z)+\alpha zq'(z)\\ &\prec \frac{z(f*\Phi)'(z)}{(f*\Psi)(z)}\left[\beta+\alpha\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)}+\alpha\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}-\alpha\frac{z(f*\Psi)'(z)}{(f*\Psi)(z)}\right],\end{aligned}$$

then

$$q(z) \prec \frac{z(f*\Phi)'(z)}{(f*\Psi)(z)}$$

and q *is the best subordinant.*

By fixing $\Phi(z)=\Psi(z)$ in Corollary 3.3, we obtain the following corollary.

Corollary 3.4. *Let* $\alpha \neq 0$ *and* $\beta \geq 1$*. Let* q *be convex univalent in* Δ *with* $q(0)=1$ *and* (3.1) *holds true. If* $f \in \mathcal{A}$ *and*

$$(\beta-\alpha)q(z)+\alpha q^2(z)+\alpha zq'(z) \prec \beta\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}+\alpha\frac{z^2(f*\Phi)''(z)}{(f*\Phi)(z)}\ ,$$

then

$$q(z) \prec \frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}$$

and q *is the best subordinant.*

We conclude this section by stating the following sandwich results.

Theorem 3.5. *Let* q_1 *and* q_2 *be convex univalent in* Δ, $\alpha \neq 0$ *and* $\beta \geq 1$*. Suppose* q_2 *satisfies* (2.3) *and* q_1 *satisfies* (3.1)*. Moreover, suppose*

$$\frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \in \mathcal{H}[1,1]\cap Q$$

and $\Upsilon(f,\Phi,\Psi,\alpha,\beta)$ *is univalent in* Δ*. If* $f \in \mathcal{A}$ *satisfies*

$$\begin{aligned}(\beta-\alpha)q_1(z)+\alpha q_1^2(z)+\alpha zq_1'(z) &\prec \Upsilon(f,\Phi,\Psi,\alpha,\beta)\\ &\prec (\beta-\alpha)q_2(z)+\alpha q_2^2(z)+\alpha zq_2'(z),\end{aligned}$$

where $\Upsilon(f,\Phi,\Psi,\alpha,\beta)$ *is given by* (2.5)*, then*

$$q_1(z) \prec \frac{H_m^l[\alpha_1+1](f*\Phi)(z)}{H_m^l[\alpha_1](f*\Psi)(z)} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By making use of Corollaries 2.5 and 3.2, we state the following corollary.

Corollary 3.6. *Let q_1 and q_2 be convex univalent in Δ, $\alpha \neq 0$ and $\beta \geq 1$. Suppose q_2 satisfies* (2.3) *and q_1 satisfies* (3.1)*. Moreover, suppose*

$$\frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)} \in \mathcal{H}[1,1] \cap Q$$

and $\Upsilon_1(f,\Phi,\Psi,\alpha,\beta)$ is univalent in Δ. If $f \in \mathcal{A}$ satisfies

$$\begin{aligned}(\beta-\alpha)q_1(z)+\alpha q_1^2(z)+\alpha z q_1'(z) &\prec \Upsilon_1(f,\Phi,\Psi,\alpha,\beta)\\ &\prec (\beta-\alpha)q_2(z)+\alpha q_2^2(z)+\alpha z q_2'(z),\end{aligned}$$

where $\Upsilon_1(f,\Phi,\Psi,\alpha,\beta)$ is given by (2.9)*, then*

$$q_1(z) \prec \frac{L(a+1,c)(f*\Phi)(z)}{L(a,c)(f*\Psi)(z)} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and best dominant.

By making use of Corollaries 2.7 and 3.3, we state the following result.

Corollary 3.7. *Let q_1 and q_2 be convex univalent in Δ, $\alpha \neq 0$ and $\beta \geq 1$. Suppose q_2 satisfies* (2.3) *and q_1 satisfies* (3.1)*. Moreover, suppose*

$$\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)} \in \mathcal{H}[1,1] \cap Q$$

and

$$\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)}\left[\beta+\alpha\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)}+\alpha\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}-\alpha\frac{z(f*\Psi)'(z)}{(f*\Psi)(z)}\right]$$

is univalent in Δ. If $f \in \mathcal{A}$ satisfies

$$\begin{aligned}&(\beta-\alpha)q_1(z)+\alpha q_1^2(z)+\alpha z q_1'(z)\\ &\prec \frac{z(f*\Phi)'(z)}{(f*\Psi)(z)}\left[\beta+\alpha\frac{z(f*\Phi)'(z)}{(f*\Psi)(z)}+\alpha\frac{z(f*\Phi)''(z)}{(f*\Phi)'(z)}-\alpha\frac{z(f*\Psi)'(z)}{(f*\Psi)(z)}\right]\\ &\prec (\beta-\alpha)q_2(z)+\alpha q_2^2(z)+\alpha z q_2'(z),\end{aligned}$$

then

$$q_1(z) \prec \frac{z(f*\Phi)'(z)}{(f*\Psi)(z)} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and best dominant.

By making use of Corollaries 2.6 and 3.4, we get the following result.

Corollary 3.8. *Let q_1 and q_2 be convex univalent in Δ, $\alpha \neq 0$ and $\beta \geq 1$. Suppose q_2 satisfies* (2.3) *and q_1 satisfies* (3.1)*. Moreover, suppose*

$$\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} \in \mathcal{H}[1,1] \cap Q$$

and

$$\beta\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}+\alpha\frac{z^2(f*\Phi)''(z)}{(f*\Phi)(z)}$$

is univalent in Δ. *If* $f\in\mathcal{A}$ *satisfies*

$$\begin{aligned}(\beta-\alpha)q_1(z)+\alpha q_1^2(z)+\alpha zq_1'(z)&\\ \prec\frac{\beta z(f*\Phi)'(z)}{(f*\Phi)(z)}&+\frac{\alpha z^2(f*\Phi)''(z)}{(f*\Phi)(z)}\\ \prec(\beta-\alpha)q_2(z)&+\alpha q_2^2(z)+\alpha zq_2'(z),\end{aligned}$$

then

$$q_1(z)\prec\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)}\prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

Acknowledgement. The authors would like to thank the referee for his valuable suggestions.

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Received May 18, 2007