ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXI, 2007 SECTIO A 129–135

MAGDALENA SOBCZAK-KNEĆ

Koebe domains for certain subclasses of starlike functions

Abstract. The Koebe domain's problem in the class of starlike functions with real coefficients was considered by M. T. McGregor [3]. In this paper we determined the Koebe domain for the class of starlike functions with real coefficients and the fixed second coefficient.

1. Introduction. Let S^* denote the class of analytic and univalent functions f in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = f'(0) - 1 = 0$ and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \qquad z \in \Delta.$$

The class S^* is called the class of starlike functions.

In this paper we will study a subclass of the class S^*, i.e. the class S^*R which contains the starlike functions with real coefficients. In 1964 M. T. McGregor [3] found the set $\bigcap_{f \in S^*R} f(\Delta)$, which is called the Koebe domain for the class S^*R.

Theorem 1 ([3])**.** *The Koebe domain for the class S^*R is symmetric with respect to the real axis and the boundary of this domain in the upper half*

2000 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C80.

Key words and phrases. Real coefficients, Koebe domain, starlike functions, fixed second coefficient.

plane is given by the polar equation $w = \rho(\theta)e^{i\theta}$*, where*

$$\rho(\theta) = \frac{1}{4}\left(\frac{\theta}{\pi}\right)^{-\frac{\theta}{\pi}}\left(1-\frac{\theta}{\pi}\right)^{\frac{\theta}{\pi}-1}, \qquad \theta \in [0,\pi]. \tag{1}$$

The extremal functions are of the form

$$F_\theta(z) = \frac{z}{(1-z)^{\frac{2\theta}{\pi}}(1+z)^{2(1-\frac{\theta}{\pi})}}, \quad z \in \Delta, \quad \theta \in [0,\pi].$$

2. Main Results.

Theorem 2. *If* $f \in S^*R$ *and* $\rho e^{i\theta} \notin f(\Delta)$*, then* $f \prec M\cdot F_\theta$*, where* $M = \frac{\rho}{\rho(\theta)}$*,* $\theta \in [0,\pi]$ *and* $\rho(\theta)$ *is given by* (1)*.*

Proof. Let $f \in S^*R$ and $\rho e^{i\theta} \notin f(\Delta)$. Since $f \in S^*R$, it means that f does not admit values, which are on the rays l and $\bar{l}$, where

$$l : \{\zeta \in \mathbb{C} : \zeta = \rho e^{i\theta}t, t \geq 1\}, \quad \bar{l} : \{\bar{\zeta} : \zeta \in l\}.$$

The function

$$\frac{\rho}{\rho(\theta)}F_\theta$$

maps the unit disk Δ onto the plane $\mathbb{C}$ without the rays l and $\bar{l}$. Moreover, $f \in S^*R$, so

$$f(\Delta) \subset \frac{\rho}{\rho(\theta)}F_\theta(\Delta).$$

From the above as well as from the univalence of F_θ we conclude that $f \prec M \cdot F_\theta$, where $M = \frac{\rho}{\rho(\theta)}$, $\theta \in [0,\pi]$. □

Remark 1. Theorem 1 results from Theorem 2. We have

$$f \prec M \cdot F_\theta.$$

Hence

$$1 = f'(0) \leq M \cdot F_\theta'(0).$$

This condition is equivalent to $M \geq 1$.

Let $f = z + a_2z^2 + \cdots \in S^*R$ and $\rho e^{i\theta} \notin f(\Delta)$. In the next theorem we determine the region of values (ρ, a_2) for a fixed $\theta \in [0, 2\pi]$. In this research we can discuss only $\theta \in [0,\pi]$, because the region of values (ρ, a_2) is symmetric with respect to the real axis.

Theorem 3. *If* $f = z + a_2z^2 + \cdots \in S^*R$ *and* $\rho e^{i\theta} \notin f(\Delta)$*, then for a fixed* $\theta \in [0,\pi]$*, the region of values* (ρ, a_2) *is of the form*

$$A_{\rho,a_2} := \left\{(\rho,a_2) : \ \frac{1}{\rho}\left(\frac{\pi}{\theta}-1\right)^{\frac{\theta}{\pi}-1} - 2 \leq a_2 \leq 2 - \frac{1}{\rho}\left(\frac{\pi}{\theta}-1\right)^{\frac{\theta}{\pi}}\right\}.$$

Proof. Let $f \in S^*R$ and $\rho e^{i\theta} \notin f(\Delta)$. From Theorem 2 and [2] we have

$$f(z) = M \cdot F_\theta\left(\frac{h(z)}{M}\right),$$

where $M = \frac{\rho}{\rho(\theta)} \geq 1$. The function $h(z)$ is univalent, with real coefficients, bounded by M and such that

$$M \cdot F_\theta\left(\frac{h(z)}{M}\right) \in S^*.$$

Denoting

$$\begin{aligned} f(z) &= z + a_2 z^2 + \dots \\ F_\theta(z) &= z + b_2(\theta) z^2 + \dots \\ h(z) &= z + c_2 z^2 + \dots . \end{aligned}$$

we have

$$a_2 = c_2 + \frac{1}{M} b_2(\theta) \qquad \text{and} \qquad b_2(\theta) = 2\left(\frac{2\theta}{\pi} - 1\right).$$

For the function $h(z)$, the following inequalities are true [1]:

$$-2\left(1 - \frac{1}{M}\right) \leq c_2 \leq 2\left(1 - \frac{1}{M}\right).$$

Hence

$$a_2 \leq 2\left(1 - \frac{1}{M}\right) + \frac{2}{M}\left(\frac{2\theta}{\pi} - 1\right),$$

and consequently

$$a_2 \leq 2 - \frac{1}{\rho}\left(\frac{\pi}{\theta} - 1\right)^{\frac{\theta}{\pi}}.$$

Moreover,

$$a_2 \geq -2\left(1 - \frac{1}{M}\right) + \frac{2}{M}\left(\frac{2\theta}{\pi} - 1\right),$$

and

$$a_2 \geq -2 + \frac{1}{\rho}\left(\frac{\pi}{\theta} - 1\right)^{\frac{\theta}{\pi} - 1}.$$

Then we have

$$\frac{1}{\rho}\left(\frac{\pi}{\theta} - 1\right)^{\frac{\theta}{\pi} - 1} - 2 \leq a_2 \leq 2 - \frac{1}{\rho}\left(\frac{\pi}{\theta} - 1\right)^{\frac{\theta}{\pi}}.$$

We shall prove, that for the fixed $\theta \in [0, \pi]$ and $\rho > \rho(\theta)$ there are functions $f \in S^*R$, $\rho e^{i\theta} \notin f(\Delta)$ such that $\frac{f''(0)}{2!}$ assumes all values from the range

$$\left[\frac{1}{\rho}\left(\frac{\pi}{\theta} - 1\right)^{\frac{\theta}{\pi} - 1} - 2,\, 2 - \frac{1}{\rho}\left(\frac{\pi}{\theta} - 1\right)^{\frac{\theta}{\pi}}\right].$$

We consider the univalent functions

$$w = f_{M,t}(z), \quad f_{M,t}(z) = z + c_2(t)z^2 + \dots$$

for which the following equation is satisfied

$$\frac{z}{1 - 2tz + z^2} = \frac{w}{1 - 2t\frac{w}{M} + \frac{w^2}{M^2}}.$$

These functions map the unit disk Δ on the disk $|w| < M$ with one or two slits on the real axis. Their coefficients $c_2(t) = 2t(1 - \frac{1}{M})$, $t \in [-1, 1]$, assume all values from the range $[-2(1-\frac{1}{M}), 2(1-\frac{1}{M})]$. Since the functions

$$f(z) = M \cdot F_\theta\left(\frac{h(z)}{M}\right) = z + a_2(t)z^2 + \dots, \text{ where } h(z) = f_{M,t}(z),$$

are starlike, $\rho e^{i\theta} \notin f(\Delta)$, therefore $a_2(t)$ assumes all values from the range $\left[\frac{1}{\rho}\left(\frac{\pi}{\theta} - 1\right)^{\frac{\theta}{\pi}-1} - 2,\, 2 - \frac{1}{\rho}\left(\frac{\pi}{\theta} - 1\right)^{\frac{\theta}{\pi}}\right]$. □

On figure 1 there is the set A_{ρ,a_2} for fixed θ.

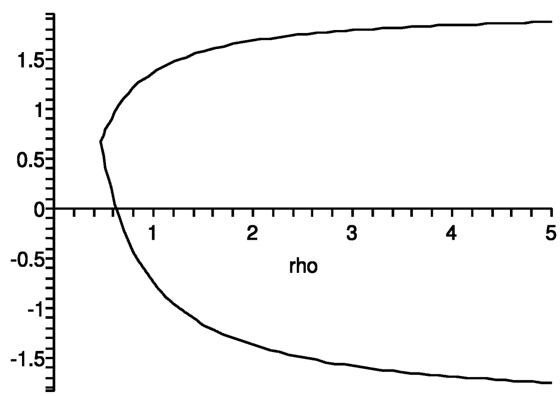

FIGURE 1. The set A_{ρ,a_2} for $\theta = \frac{2}{3}\pi$.

Definition 1. We say, that the function f is in S_a^* if $f \in S^*$ and $\frac{1}{2}f''(0) = a$, $a \geq 0$ i.e.

$$S_a^* = \{f \in S^* : f(z) = z + az^2 + \dots\}.$$

Rogosinski in paper [4] determined the Koebe domain for the class S_a^*.

Theorem 4. *The Koebe domain for the class S_a^*, $a \in [0, 2)$, is symmetric with respect to the real axis and the boundary of this domain in the upper half plane is given by the polar equation $w = \rho(\theta)e^{i\theta}$, where*

$$\rho(\theta) = \frac{2 + a\cos\theta}{4 - a^2}, \quad a \geq 0, \; \theta \in [0, \pi].$$

We determine the Koebe domain for the class S_a^*R consisting of the functions from the class S_a^* which have real coefficients. From Theorem 3 we conclude the following theorem for the class S_a^*R.

Theorem 5. *The Koebe domain for the class S_a^*R is symmetric with respect to the real axis and the boundary of this domain in the upper half plane is given by the polar equation $w = \rho_a(\theta)e^{i\theta}$, where*

$$\rho_a(\theta) = \begin{cases} \frac{1}{2-a}\left(\frac{\pi}{\theta}-1\right)^{\frac{\theta}{\pi}}, & \theta \in \left[0, \frac{(2+a)\pi}{4}\right], \\ \frac{1}{2+a}\left(\frac{\pi}{\theta}-1\right)^{\frac{\theta}{\pi}-1}, & \theta \in \left(\frac{(2+a)\pi}{4}, \pi\right]. \end{cases} \tag{2}$$

Proof. Let $a_2 = a$. From Theorem 3 we have

$$\rho \geq \frac{1}{2-a}\left(\frac{\pi}{\theta}-1\right)^{\frac{\theta}{\pi}}, \quad \text{where} \quad \theta \in \left[0, \frac{(2+a)\pi}{4}\right]$$

and

$$\rho \geq \frac{1}{2+a}\left(\frac{\pi}{\theta}-1\right)^{\frac{\theta}{\pi}-1}, \quad \text{where} \quad \theta \in \left(\frac{(2+a)\pi}{4}, \pi\right]. \qquad \square$$

On figures 2, 3, 4 there are the Koebe domains for the class S_a^*R for some fixed $a_2 = a$.

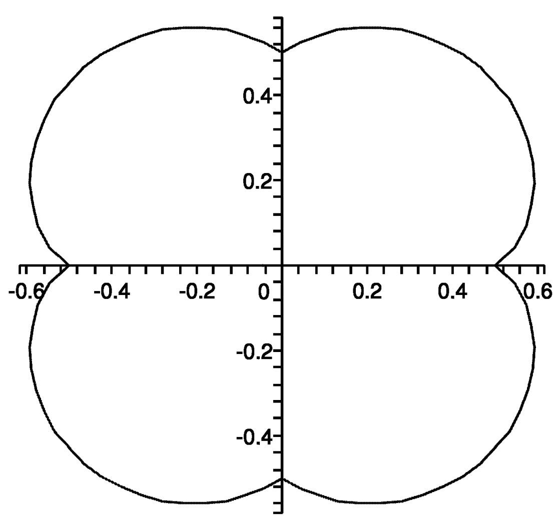

FIGURE 2. The Koebe domain for the class S_a^*R, $a = 0$.

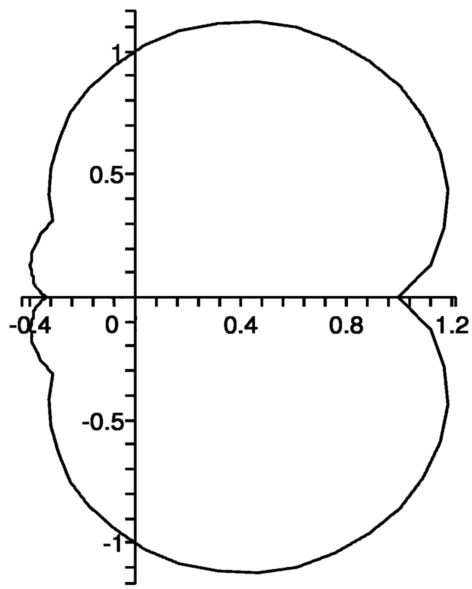

FIGURE 3. The Koebe domain for the class S_a^*R, $a = 1$.

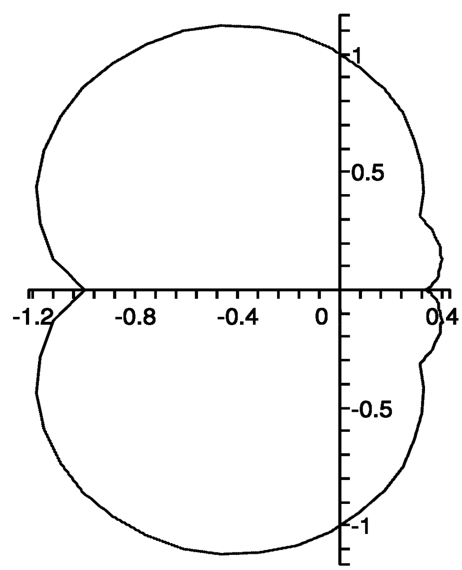

FIGURE 4. The Koebe domain for the class S_a^*R, $a = -1$.

Definition 2. We say that the function $f(z)$ is n-symmetric function in Δ, if for fixed $z \in \Delta$ the following condition is satisfied

$$f\left(e^{\frac{2\pi i}{n}} z\right) = e^{\frac{2\pi i}{n}} f(z).$$

We say that the set D is n-symmetric, if the set satisfies the condition $e^{\frac{2\pi i}{n}} D = D$. The set λD is understood as $\{\lambda z : z \in D\}$.

We denote by S^*R^n the class of starlike and n-symmetric functions with real coefficients. From Theorem 5 we have

Corollary 1. *The Koebe domain for the class S^*R^n with fixed $a_{n+1} = b$, $n \geq 2$ is n-symmetric, symmetric with respect to the real axis and the line $\zeta = e^{\frac{\pi i}{n}} t$ and the boundary of this domain in the set $\left\{\zeta \in \mathbb{C} : 0 \leq \arg \zeta \leq \frac{\pi}{n}\right\}$ is given by the polar equation $w = \rho_{b,n}(\theta) e^{i\theta}$ where*

$$\rho_{b,n}(\theta) = \sqrt[n]{\rho_a(n\theta)}, \quad a = bn,\ 0 \leq \theta \leq \frac{\pi}{n}.$$

Proof. For the function $f \in S^*R^n$ the following condition is satisfied

$$f \in S^*_a R \iff g \in S^*R^n,\ \frac{g^{(n+1)}(0)}{(n+1)!} = \frac{a_2}{n}, \tag{3}$$

where $g(z) = \sqrt[n]{f(z^n)}$. Let $b = \frac{a_2}{n}$. We determine the set of the form $\bigcap_{S^*R^n} g(\Delta)$. From Theorem 5 we know that the boundary of the Koebe domain in the class S^*R is of the form $w = \rho_a(\theta) e^{i\theta}$ where $\rho_a(\theta)$ is given by (2). From (3) we have

$$\sqrt[n]{w} = \sqrt[n]{\rho_a(t)} e^{\frac{it}{n}}, \quad t \in [0, \pi],$$

and consequently for $a = bn$, $\theta = \frac{t}{n} \in [0, \frac{\pi}{n}]$ we have

$$\sqrt[n]{w} = \sqrt[n]{\rho_{bn}(n\theta)} e^{i\theta}. \qquad \square$$

References

[1] Goodman, A. W., *Univalent Functions*, Mariner Pub. Co., Tampa, 1983.

[2] Koczan, L., *On classes generated by bounded functions*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A **52**, No. 2 (1998), 95–101.

[3] McGregor, M. T., *On three classes of univalent functions with real coefficients*, J. London Math. Soc. **39** (1964), 43–50.

[4] Rogosinski, W. W., *On subordinate functions*, Proc. Camb. Phil. Soc. **35** (1939), 1–26.

Magdalena Sobczak-Kneć
Department of Applied Mathematics
Lublin University of Technology
ul. Nadbystrzycka 38D
20-618 Lublin, Poland
e-mail: m.sobczak-knec@pollub.pl

Received May 24, 2007