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Inequalities for Bergman spaces

ABSTRACT. In this paper we prove an inequality for weighted Bergman spaces A^p_α, $0 < p < \infty$, $-1 < \alpha < \infty$, that corresponds to Hardy–Littlewood inequality for Hardy spaces. We give also a necessary and sufficient condition for an analytic function f in $\mathbb{D}$ to belong to A^p_α.

1. Introduction and statement of results. Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. For $0 < p < \infty$ the Hardy space H^p consists of analytic functions f in $\mathbb{D}$ such that

$$||f||_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty,$$

where

$$M_p(r,f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}.$$

For $-1 < \alpha < \infty$ and $0 < p < \infty$ the weighted Bergman space A^p_α consists of analytic functions f in $\mathbb{D}$ such that

$$||f||^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty,$$

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where

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$$

and $dA(z)$ is the area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$.

In this paper we obtain some inequalities for Bergman spaces that correspond to the inequalities for Hardy spaces. In the proof of Theorem 2 we use the general method for translating the known equalities for H^p spaces to Bergman spaces version described in [9]. We first recall the Hardy–Littlewood inequality for H^p spaces.

Theorem HL. *Suppose that* $0 < p < q \leq \infty$, $\beta = \frac{1}{p} - \frac{1}{q}$, $l \geq q$. *Then there is a positive constant* C *such that*

$$\int_0^1 (1-r)^{l\beta-1} M_q^l(r,f)dr \leq C||f||_{H^p}^l.$$

Here we prove the following theorem for Bergman spaces.

Theorem 1. *Suppose that* $0 < p < q \leq \infty$, $l \geq p$, $\beta = \frac{2+\alpha}{p} - \frac{1}{q}$, $-1 < \alpha < \infty$. *Then there exists a positive constant* C *such that*

$$\int_0^1 (1-r)^{l\beta-1} M_q^l(r,f)dr \leq C||f||_{A_\alpha^p}^l.$$

We note that Theorem 1 generalizes Lemma 5 in [8]. In 1988 D. Luecking proved the following generalization of the Littlewood and Paley inequality for Hardy spaces.

Theorem L. *Let* $0 < p, s < +\infty$. *Then there exists a constant* $C = C(p,s)$ *such that*

$$\int_{\mathbb{D}} |f(z)|^{p-s}|f'(z)|^s(1-|z|)^{s-1}dA(z) \leq C||f||_{H^p}^p \tag{1}$$

for all $f \in H^p$ *if and only if* $2 \leq s < p+2$.

For Bergman spaces we get

Theorem 2. *Let* $0 < p < \infty$, $-1 < \alpha < \infty$ *and* $0 \leq s < p+2$. *Then there exists a constant* $C = C(p,s)$ *such that*

$$\int_{\mathbb{D}} |f(z)|^{p-s}|f'(z)|^s(1-|z|)^s dA_\alpha(z) \leq C||f||_{A_\alpha^p}^p \tag{2}$$

for all $f \in A_\alpha^p$.

The next theorem is, in some sense, the converse of Theorem 2.

Theorem 3. *Suppose that* $0 < p < \infty$, $s \in \mathbb{R}$, $\alpha > -1$ *and* $f \in H(\mathbb{D})$ *with* $f(0) = 0$. *Then there exists a constant* $C = C(p,s)$ *such that*

$$\int_{\mathbb{D}} |f(z)|^{p-s}|f'(z)|^s \left(\log\frac{1}{|z|}\right)^s dA_\alpha(z) \geq C||f||_{A_\alpha^p}^p\ . \tag{3}$$

Corollary. *Let $f \in Hol(\mathbb{D})$ with $f(0) = 0$, $0 < p < \infty$, $-1 < \alpha < \infty$ and $0 \le s < p + 2$. Then the following conditions are equivalent:*

i) $f \in A^p_\alpha$,

ii) $\displaystyle\int_{\mathbb{D}} |f(z)^{p-s}|f'(z)^s(1-|z|^2)^s dA_\alpha(z) < \infty.$

2. Proofs. For positive functions f, g defined in $\mathbb{D}$ we write

$$f(z) \sim g(z) \quad \text{as} \quad |z| \to 1^-,$$

if

$$\lim_{|z|\to 1^-} \frac{f(z)}{g(z)} = K \in (0, +\infty).$$

We will use the following well-known lemma. Its proof can be found e.g. in [9, p. 15].

Lemma. *For any $\beta > 0$*

$$\int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{1+\beta}} \sim \frac{1}{(1-|z|^2)^\beta} \quad as \quad |z| \to 1^-.$$

Proof of Theorem 1. It follows from the proof of Theorem 5.9 in [2] that for every analytic function in $\mathbb{D}$, $r < 1$, $0 < p < q \le \infty$,

$$M_q(r, f) \le (1-r)^{\frac{1}{q}-\frac{1}{p}} M_p(r, f).$$

Furthermore, by the monotonicity of the integral mean $M^p_p(r, f)$ we get

$$\begin{aligned} ||f||^p_{A^p_\alpha} &= C\int_0^1 M^p_p(t, f)\big(1-t^2\big)^\alpha dt \ge C\int_r^1 M^p_p(t, f)(1-t)^\alpha dt \\ &\ge CM^p_p(r, f)\int_r^1 (1-t)^\alpha dt = CM^p_p(r, f)(1-r)^{\alpha+1}, \end{aligned}$$

which implies

$$M_p(r, f) \le C\frac{||f||_{A^p_\alpha}}{(1-r)^{\frac{\alpha+1}{p}}}, \quad 0 < r < 1.$$

Therefore

$$\begin{aligned} M^l_q(r, f)(1-r)^{l\beta-1-\alpha} &\le M^{l-p}_p(r, f)(1-r)^{\left(\frac{1}{q}-\frac{1}{p}\right)l+l\beta-1-\alpha} M^p_p(r, f) \\ &\le C||f||^{l-p}_{A^p_\alpha}(1-r)^{-\frac{1}{p}(1+\alpha)(l-p)}(1-r)^{\left(\frac{1}{q}-\frac{1}{p}\right)l+l\beta-1-\alpha} M^p_p(r, f) \\ &= C||f||^{l-p}_{A^p_\alpha}(1-r)^{l\left(\beta-(\frac{2+\alpha}{p}-\frac{1}{q})\right)} M^p_p(r, f) \\ &\le C||f||^{l-p}_{A^p_\alpha} M^p_p(r, f). \end{aligned}$$

Multiplying both sides by $(1-r)^\alpha$ and integrating with respect to r give

$$\int_0^1 (1-r)^{l\beta-1} M^l_q(r, f)dr \le C||f||^l_{A^p_\alpha}. \qquad \square$$

We remark that the exponent β is best possible. If $\beta < \frac{2+\alpha}{p} - \frac{1}{q}$, then $\beta = \frac{2+\alpha}{p} - \epsilon - \frac{1}{q} = \gamma - \frac{1}{q}$, where $\epsilon > 0$. Thus the function $f(z) = (1-z)^{-\gamma} \in A^p_\alpha$ and by Lemma,

$$\begin{aligned}\int_0^1 (1-r)^{l\beta-1} M_q^l(r,f)dr &\geq C\int_0^1 (1-r)^{l\beta-1}(1-r)^{-(\gamma q-1)\frac{l}{q}}dr \\ &= C\int_0^1 (1-r)^{l\left(\beta-\gamma+\frac{1}{q}\right)-1}dr = +\infty.\end{aligned}$$

Proof of Theorem 2. Assume first that $f \in A^p_\alpha$ and $2 \leq s < p+2$. In this case the method described in [9] can be applied. By Theorem L

$$\int_{\mathbb{D}} |f_r(z)|^{p-s}|f_r'(z)|^s(1-|z|)^{s-1}dA(z) \leq ||f_r||^p_{H^p}, \tag{4}$$

where $f_r(z) = f(rz)$, $0 < r < 1$, $z \in \mathbb{D}$. The left-hand side of inequality (4) is equal to

$$\begin{aligned}&\int_{\mathbb{D}} |f(rz)|^{p-s}|f'(rz)|^s r^s(1-|z|)^{s-1}dA(z) \\ &= \int_{|\zeta|<r} |f(\zeta)|^{p-s}|f'(\zeta)|^s r^s\left(1-\frac{|\zeta|}{r}\right)^{s-1} r^{-2}dA(\zeta) \\ &= \frac{1}{\pi}\int_0^r\int_0^{2\pi} |f(\rho e^{i\theta})|^{p-s}|f'(\rho e^{i\theta})|^s\left(1-\frac{\rho}{r}\right)^{s-1} r^{s-2}\rho d\theta d\rho.\end{aligned} \tag{5}$$

Multiplying both sides of (4) by $(1+\alpha)2r(1-r^2)^\alpha$ and integrating with respect to r we get

$$\begin{aligned}\frac{2}{\pi}\int_0^1\int_0^r\int_0^{2\pi} &|f(\rho e^{i\theta})|^{p-s}|f'(\rho e^{i\theta})|^s\left(1-\frac{\rho}{r}\right)^{s-1} r^{s-2}\rho d\theta d\rho(1+\alpha)r(1-r^2)^\alpha dr \\ &\leq \frac{1}{\pi}\int_0^1\int_0^{2\pi} |f(re^{i\theta})|^p d\theta(1+\alpha)r(1-r^2)^\alpha dr.\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}(\alpha+1)\frac{2}{\pi}\int_0^1\int_\rho^1\left(\int_0^{2\pi}|f(\rho e^{i\theta})|^{p-s}|f'(\rho e^{i\theta})|^s\rho d\theta\right)&\left(1-\frac{\rho}{r}\right)^{s-1} r^{s-1}(1-r^2)^\alpha dr d\rho \\ &\leq \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) = ||f||^p_{A^p_\alpha}.\end{aligned}$$

Put

$$F(\rho) = \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-s}|f'(\rho e^{i\theta})|^s\rho d\theta.$$

Then the left-hand side of the last inequality can be written as

$$(\alpha+1)\frac{2}{\pi}\int_0^1\int_\rho^1 F(\rho)(r-\rho)^{s-1}(1-r^2)^\alpha dr d\rho$$
$$=(\alpha+1)\frac{2}{\pi}\int_0^1 F(\rho)\left(\int_\rho^1 (r-\rho)^{s-1}(1-r^2)^\alpha dr\right)d\rho.$$

Now, since

$$\int_\rho^1 (r-\rho)^{s-1}(1-r^2)^\alpha dr \ge \int_{\frac{1+\rho}{2}}^1 (r-\rho)^{s-1}(1-r^2)^\alpha dr$$
$$\ge \int_{\frac{1+\rho}{2}}^1 (1-r)^{s+\alpha-1}dr = \frac{1}{(s+\alpha)2^{s+\alpha}}(1-\rho)^{s+\alpha},$$

we get

$$(\alpha+1)\frac{2}{\pi}\int_0^1 F(\rho)\left(\int_\rho^1 (r-\rho)^{s-1}(1-r^2)^\alpha dr\right)d\rho$$
$$\ge \frac{(\alpha+1)}{(s+\alpha)2^{s+\alpha}}\frac{2}{\pi}\int_0^1 F(\rho)(1-\rho)^{s+\alpha}d\rho$$
$$=\frac{(\alpha+1)}{(s+\alpha)2^{s+\alpha}}\frac{2}{\pi}\int_0^1\int_0^{2\pi}|f(\rho e^{i\theta})|^{p-s}|f'(\rho e^{i\theta})|^s(1-\rho)^{s+\alpha}\rho d\theta d\rho$$
$$=\frac{1}{(s+\alpha)2^{s+\alpha-1}}\int_{\mathbb{D}}|f(z)|^{p-s}|f'(z)|^s(1-|z|)^s dA_\alpha(z).$$

Suppose now that $0<s<2$. Then using Hölder's inequality we obtain

$$\int_{\mathbb{D}}|f(z)|^{p-s}|f'(z)|^s(1-|z|)^s dA_\alpha(z)$$
$$=\int_{\mathbb{D}}|f(z)|^{\frac{(p-2)s}{2}}|f'(z)|^s(1-|z|)^s|f(z)|^{\frac{(2-s)p}{2}}dA_\alpha(z)$$
$$\le\left\{\int_{\mathbb{D}}|f(z)|^{p-2}|f'(z)|^2(1-|z|)^2 dA_\alpha(z)\right\}^{\frac{s}{2}}\left\{\int_{\mathbb{D}}|f(z)|^{\frac{(2-s)p}{2}\frac{2}{2-s}}dA_\alpha(z)\right\}^{\frac{2-s}{2}}$$
$$\le\left\{(2+\alpha)2^{1+\alpha}||f||^p_{A^p_\alpha}\right\}^{\frac{s}{2}}\left\{||f||^p_{A^p_\alpha}\right\}^{1-\frac{s}{2}}=C||f||^p_{A^p_\alpha},$$

where the last inequality follows from the case $s=2$. □

We remark that the function $f(z)=z$ gives the estimate $s>-\alpha-1$. This example shows also that the inequality does not hold for $s=p+2$. We do not know whether the condition $s\ge 0$ is best possible.

In the proof of the next theorem we use the following version of Hölder's inequality (see e.g., [4, p. 140]). Suppose that F i G are nonnegative and $F\in(L^p,d\mu)$, $G\in(L^q,d\mu)$. For $p\ne 0$ let q be its conjugate, that is,

$\frac{1}{p}+\frac{1}{q}=1$. If $p\in(0,1)$ or $p<0$, then

$$\int_X FG\,d\mu \ge \left\{\int_X F^p d\mu\right\}^{\frac{1}{p}}\left\{\int_X G^q d\mu\right\}^{\frac{1}{q}}. \tag{6}$$

Proof of Theorem 3. Proceeding as in the proof of Theorem 2 in [6] one can get

$$\begin{aligned}&\int_{\mathbb{D}}|f(z)|^p dA_\alpha(z)\\ &\qquad = p^2(1+\alpha)\int_0^1 r(1-r^2)^\alpha\int_0^r\int_{|z|<t}\frac{1}{t}|f(z)|^{p-2}|f'(z)|^2 dA(z)dtdr.\end{aligned}$$

By Fubini's theorem the right-hand side of the last inequality is equal to

$$\begin{aligned}&\frac{p^2}{2}\int_0^1\frac{(1-t^2)^{\alpha+1}}{t}\int_{|z|<\ t}|f(z)|^{p-2}|f'(z)|^2 dA(z)dt\\ &\quad\le\frac{p^2}{2}\int_{\mathbb{D}}\int_{|z|}^1\frac{(1-t^2)^{\alpha+1}}{t}dt|f(z)|^{p-2}|f'(z)|^2 dA(z)\\ &\quad\le\frac{p^2}{2}\int_{\mathbb{D}}\int_{|z|}^1\frac{(1-|z|^2)^{\alpha+1}}{t}dt|f(z)|^{p-2}|f'(z)|^2 dA(z)\\ &\quad\le\frac{p^2}{2(\alpha+1)}\int_{\mathbb{D}}|f(z)|^{p-2}|f'(z)|^2(1-|z|^2)^\alpha(1-|z|^2)\log\frac{1}{|z|}dA(z)\\ &\quad\le\frac{p^2}{2(\alpha+1)}\int_{\mathbb{D}}|f(z)|^{p-2}|f'(z)|^2\left(\log\frac{1}{|z|}\right)^2 dA_\alpha(z).\end{aligned}$$

Consequently

$$\int_{\mathbb{D}}|f(z)|^p dA_\alpha(z)\le\frac{p^2}{2(\alpha+1)}\int_{\mathbb{D}}|f(z)|^{p-2}|f'(z)|^2\left(\log\frac{1}{|z|}\right)^2 dA_\alpha(z).$$

Suppose now that $s>2$ or $s<0$. Then, by Hölder's inequality (6) and the case $s=2$

$$\begin{aligned}&\int_{\mathbb{D}}|f(z)|^{p-s}|f'(z)|^s\log^s\frac{1}{|z|}dA_\alpha(z)\\ &=\int_{\mathbb{D}}|f(z)|^{\frac{(p-2)s}{2}}|f'(z)|^s\log^s\frac{1}{|z|}|f(z)|^{\frac{(2-s)p}{2}}dA_\alpha(z)\\ &\ge\left\{\int_{\mathbb{D}}|f(z)|^{p-2}|f'(z)|^2\log^2\frac{1}{|z|}dA_\alpha(z)\right\}^{\frac{s}{2}}\left\{\int_{\mathbb{D}}|f(z)|^{\frac{(2-s)p}{2}\frac{2}{2-s}}dA_\alpha(z)\right\}^{\frac{2-s}{2}}\\ &\ge C\left\{||f||^p_{A^p_\alpha}\right\}^{\frac{s}{2}}\left\{||f||^p_{A^p_\alpha}\right\}^{1-\frac{s}{2}}=C||f||^p_{A^p_\alpha}.\end{aligned}$$

Finally, assume that $0 < s < 2$. Applying (6) with $p = \frac{s}{2}$, $q = \frac{s}{s-2}$, we get

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z)|^{p-s}|f'(z)|^s \log^s \frac{1}{|z|} dA_\alpha(z) \\
&\quad = \int_{\mathbb{D}} |f(z)|^{\frac{(p-2)2}{s}} |f'(z)|^{\frac{4}{s}} \log^{\frac{4}{s}} \frac{1}{|z|} \\
&\qquad \times |f(z)|^{\frac{(s-2)(p-(s+2))}{s}} |f'(z)|^{\frac{(s+2)(s-2)}{s}} \log^{\frac{(s+2)(s-2)}{s}} \frac{1}{|z|} dA_\alpha(z) \\
&\quad \geq \left\{ \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \log^2 \frac{1}{|z|} dA_\alpha(z) \right\}^{\frac{2}{s}} \\
&\qquad \times \left\{ \int_{\mathbb{D}} |f(z)|^{p-(s+2)}|f'(z)|^{s+2} \log^{s+2} \frac{1}{|z|} dA_\alpha(z) \right\}^{\frac{s-2}{s}} \\
&\quad \geq C \left\{ ||f||^p_{A^p_\alpha} \right\}^{\frac{s}{2}} \left\{ ||f||^p_{A^p_\alpha} \right\}^{1-\frac{s}{2}} = C||f||^p_{A^p_\alpha},
\end{aligned}$$

where the last inequality follows from the proved case. □

References

[1] Beatrous, F. Jr., Burbea, J., *Characterizations of spaces of holomorphic functions in the ball*, Kodai Math. J. **8** (1985), 36–51.

[2] Duren, P. L., *Theory of H^p Spaces*, Academic Press, New York–London, 1970.

[3] Hardy, G. H., Littlewood, J. E., *Some properties of fractional integrals II*, Math. Z. **34** (1932), 403–439.

[4] Hardy, G. H., Littlewood, J. E. and Pólia, G., *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1952.

[5] Luecking, D. H., *A new proof of an inequality of Littelwood and Paley*, Proc. Amer. Math. Soc. **103** (1988), 887–983.

[6] Nowak, M., *Bloch space on the unit ball of C^n*, Ann. Acad. Sci. Fenn. Math. **23** (1998), 461–473.

[7] Ouyang, C., Yang, W. and Zhao, R., *Characterization of Bergman spaces and Bloch spaces in the unit ball*, Trans. Amer. Math. Soc. **347** (1995), 4301–4313.

[8] Watanabe, H., *Some properties of functions in Bergman space A^p*, Proc. Fac. Sci. Tokai Univ. **13** (1978), 39–54.

[9] Zhu, K., *Translating inequalities between Hardy and Bergman spaces*, Amer. Math. Monthly **111** (2004), 520–525.

[10] Zhu, K., *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York, 2005.

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