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## Differential sandwich theorems for multivalent functions

ABSTRACT. In the present paper, we apply methods based on differential subordinations and superordinations in order to derive several subordination results for multivalent functions involving the Hadamard product.

**1. Introduction and motivations.** Let  $\mathcal{H} = \mathcal{H}(\Delta)$  be the space of all analytic functions in the *open* unit disk  $\Delta := \{z : |z| < 1\}$ . For  $a \in \mathbb{C}$ , let  $\mathcal{H}[a, n]$  be the subclass of  $\mathcal{H}$  consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots .$$

Let  $\mathcal{A}_p$  denote the class of all *analytic* and  $p$ -valent functions  $f$  of the form

$$(1.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (z \in \Delta),$$

and  $\mathcal{A} := \mathcal{A}_1$ , where  $p \in \mathbb{N} := \{1, 2, 3, \dots\}$ .

For any two analytic functions,  $f$  given by (1.1) and  $g$  given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

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their Hadamard product (or convolution) is the function  $f * g$  defined by

$$(f * g)(z) := z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

For various choices of  $g$  we get different operators; for example,

(1) For

$$g(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \cdots (\alpha_l)_{n-p}}{(\beta_1)_{n-p} \cdots (\beta_m)_{n-p}} \frac{z^n}{(n-p)!},$$

we get the Dziok–Srivastava operator

$$\Lambda(\alpha_1, \alpha_2, \dots, \alpha_l; \beta_1, \beta_2, \dots, \beta_m; z)f(z) \equiv H_{l,m}^p f(z) := (f * g)(z),$$

introduced by Dziok and Srivastava [7]; where  $\alpha_1, \alpha_2, \dots, \alpha_l, \beta_1, \beta_2, \dots, \beta_m$  are complex parameters,  $\beta_j \notin \{0, -1, -2, \dots\}$  for  $j = 1, 2, \dots, m$ ,  $l \leq m + 1$ ,  $l, m \in \mathbb{N} \cup \{0\}$ . Here  $(a)_\nu$  denotes the well-known Pochhammer symbol (or shifted factorial).

(2) For

$$g(z) = \phi_p(a, c, z) := z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \quad (c \neq 0, -1, -2, \dots),$$

we get the  $p$ -valent Carlson–Shaffer operator  $L_p(a, c)f(z) := (f * g)(z)$ . The operator

$$L(a, c)f(z) \equiv L_1(a, c)f(z) \equiv zF(a, 1; c; z) * f(z)$$

was introduced by Carlson–Shaffer [4] where  $F(a, b; c; z)$  is the Gaussian hypergeometric function.

(3) For  $g(z) = \frac{z^p}{(1-z)^{\lambda+p}}$  ( $\lambda \geq -p$ ), we obtain the  $p$ -valent Ruscheweyh operator defined by

$$D^{\lambda+p-1}f(z) := (f * g)(z) = z^p + \sum_{n=p+1}^{\infty} \binom{\lambda+n-1}{n-p} a_n z^n.$$

The operator  $D^{\lambda+p-1}f : \mathcal{A}_p \rightarrow \mathcal{A}_p$  was introduced by Patel and Cho [12]. In particular,  $D^\lambda f : \mathcal{A} \rightarrow \mathcal{A}$  for  $p = 1$  and  $\lambda \geq -1$  was introduced by Ruscheweyh [14].

(4) For

$$g(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right)^k z^n \quad (k \geq 0),$$

we get the the  $p$ -valent Sălăgean operator  $D_p^k f(z) : \mathcal{A}_p \rightarrow \mathcal{A}_p$  introduced by Shenan *et al.* [22]. In particular, the differential operator  $\mathcal{D}^k \equiv \mathcal{D}_1^k$  was initially studied by Sălăgean [15].

(5) For

$$g(z) = z^p + \sum_{n=p+1}^{\infty} n \left( \frac{n+\lambda}{p+\lambda} \right)^k z^n \quad (\lambda \geq 0; k \in \mathbb{Z}),$$

we obtain the multiplier transformation  $I_p(\lambda, k) := (f * g)(z)$  introduced by Ravichandran *et al.* [13]. In particular,  $I(\lambda, k) \equiv I_1(\lambda, k)$  was studied by Cho and Kim [5] and Cho and Srivastava [6].

(6) For

$$g(z) = z^p + \sum_{n=p+1}^{\infty} \left( \frac{n+\lambda}{1+\lambda} \right)^k z^n \quad (\lambda \geq 0; k \in \mathbb{Z}),$$

we get multiplier transformation  $J_p(\lambda, k) := (f * g)(z)$ . In particular  $J(\lambda, k) \equiv J_1(\lambda, k)$  was introduced by Cho and Srivastava [6].

With a view to recalling the principle of subordination between analytic functions, let the functions  $f$  and  $g$  be analytic in  $\Delta$ . Then we say that  $f$  is *subordinate* to  $g$ , or  $g$  is *superordinate* to  $f$ , if there exists a Schwarz function  $\omega$ , analytic in  $\Delta$  with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \Delta),$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if the function  $g$  is univalent in  $\Delta$ , the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Let  $\psi, h \in \mathcal{H}$  and let  $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \rightarrow \mathbb{C}$ . If  $p$  and  $\phi(\psi(z), z\psi'(z), z^2\psi''(z); z)$  are univalent and if  $\psi$  satisfies the second order superordination

$$(1.2) \quad h(z) \prec \phi(\psi(z), z\psi'(z), z^2\psi''(z); z),$$

then  $\psi$  is a solution of the differential superordination (1.2). An analytic function  $q$  is called a *subordinant* if  $q \prec \psi$  for all  $\psi$  satisfying (1.2). An univalent subordinant  $\tilde{q}$  that satisfies  $q \prec \tilde{q}$  for all subordinants  $q$  of (1.2) is said to be the best subordinant. Recently Miller and Mocanu [9] obtained conditions on  $h, q$  and  $\phi$  for which the following implication holds:

$$h(z) \prec \phi(\psi(z), z\psi'(z), z^2\psi''(z); z) \Rightarrow q(z) \prec \psi(z).$$

Using the results of Miller and Mocanu [9], Bulboacă [3] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [2]. Shanmugam *et al.* [17] obtained

sufficient conditions for normalized analytic functions  $f$  which satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z),$$

where  $q_1$  and  $q_2$  are given univalent functions in  $\Delta$  with  $q_1(0) = 1$  and  $q_2(0) = 1$ . On the other hand, Obradović and Owa [11] obtained subordination results for the quantity  $\left(\frac{f(z)}{z}\right)^\mu$ . A detailed investigation of starlike functions of complex order and convex functions of complex order using Briot–Bouquet differential subordination technique has been studied recently by Srivastava and Lashin [24].

In an earlier investigation, a sequence of results using differential subordination with convolution for the univalent case has been studied by Shanmugam [16]. A systematic study of the subordination and superordination using certain operators under the univalent case has been also studied by Shanmugam *et al.* [18, 19, 20]. We observe that for the multivalent case, many of the results for the operators in (1) to (6) for the multivalent case have not yet been studied.

The main object of the present sequel to the aforementioned works is to apply the methods based on the differential subordination and superordination in order to derive several subordination results for the multivalent functions involving the Hadamard product. Furthermore, as special cases, we also obtain corresponding results of Obradović and Owa [11], Shanmugam *et al.* [17], Singh [23].

In order to investigate our subordination and superordination results, we make use of the following known results.

**Definition 1** ([9, Definition 2, p. 817]). Denote by  $\mathcal{Q}$ , the set of all functions  $f$  that are analytic and injective on  $\overline{\Delta} \setminus E(f)$ , where

$$E(f) = \left\{ \zeta \in \partial\Delta : \lim_{z \rightarrow \zeta} f(z) = \infty \right\},$$

and are such that  $f'(\zeta) \neq 0$  for  $\zeta \in \partial\Delta - E(f)$ .

**Theorem A** ([8, Theorem 3.4h, p. 132]). *Let  $q$  be an univalent function in  $\Delta$  and let  $\theta$  and  $\phi$  be analytic in a domain  $D$  containing  $q(\Delta)$  with  $\phi(w) \neq 0$  when  $w \in q(\Delta)$ . Set  $Q(z) = zq'(z)\phi(q(z))$ ,  $h(z) = \theta(q(z)) + Q(z)$ . Suppose that*

- (1)  $Q$  is starlike univalent in  $\Delta$ , and
- (2)  $\Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left( \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right) > 0$  for all  $z \in \Delta$ .

If  $\psi$  is analytic in  $\Delta$ , with  $\psi(0) = q(0)$ ,  $\psi(\Delta) \subset D$  and

$$\theta(\psi(z)) + z\psi'(z)\phi(\psi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then  $\psi(z) \prec q(z)$  and  $q$  is the best dominant.

**Theorem B** ([3]). Let the function  $q$  be univalent in the unit disk  $\Delta$  and  $\vartheta$  and  $\varphi$  be analytic in a domain  $D$  containing  $q(\Delta)$ . Suppose that

- (1)  $\Re \left[ \frac{\vartheta'(q(z))}{\varphi(q(z))} \right] > 0$  for all  $z \in \Delta$ ,
- (2)  $Q(z) = zq'(z)\varphi(q(z))$  is starlike univalent in  $\Delta$ .

If  $\psi \in \mathcal{H}[q(0), 1] \cap Q$ , with  $\psi(\Delta) \subseteq D$ , and  $\vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z))$  is univalent in  $\Delta$ , and

$$(1.3) \quad \vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z)),$$

then  $q(z) \prec \psi(z)$  and  $q$  is the best subdominant.

**2. Main results.** We now prove the following result involving differential subordination between analytic functions.

**Theorem 1.** Let the function  $q$  be analytic with  $q(0) = 1$  and univalent in  $\Delta$  such that  $q(z) \neq 0$ . Let  $z \in \Delta$ ,  $\alpha, \delta, \xi, \gamma_1, \delta_1, \delta_2, \delta_3 \in \mathbb{C}$  and suppose at least one of  $\delta_1, \delta_2, \delta_3 \in \mathbb{C}$  is non-zero. Suppose  $q$  satisfies

$$(2.1) \quad \Re \left( 1 + \left( \frac{\xi q^2(z) + 2\delta q^3(z) - \gamma_1}{\delta_1 q^2(z) + \delta_2 q(z) + \delta_3} \right) - \frac{zq'(z)}{q(z)} \left( \frac{\delta_2 q(z) + 2\delta_3}{\delta_1 q^2(z) + \delta_2 q(z) + \delta_3} \right) + \frac{zq''(z)}{q'(z)} \right) > 0$$

and

$$(2.2) \quad \Re \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \left( \frac{\delta_2 q(z) + 2\delta_3}{\delta_1 q^2(z) + \delta_2 q(z) + \delta_3} \right) \right) > 0.$$

Let

$$(2.3) \quad \begin{aligned} \Psi(f, g, \mu, \xi, \beta, \delta, \gamma_1, \delta_1, \delta_3) := & \alpha + \xi \left\{ \frac{(f * g)'(z)}{pz^{p-1}} \right\}^\mu \\ & + \delta \left\{ \frac{(f * g)'(z)}{pz^{p-1}} \right\}^{2\mu} + \gamma_1 \left\{ \frac{(f * g)'(z)}{pz^{p-1}} \right\}^{-\mu} \\ & + \mu \left[ \frac{z(f * g)''(z)}{(f * g)'(z)} - (p-1) \right] \left\{ \delta_2 + \delta_1 \left\{ \frac{(f * g)'(z)}{pz^{p-1}} \right\}^\mu \right\} \\ & + \delta_3 \mu \left[ \frac{z(f * g)''(z)}{(f * g)'(z)} - (p-1) \right] \left\{ \frac{(f * g)'(z)}{pz^{p-1}} \right\}^{-\mu} \end{aligned}$$

for some  $\mu \in \mathbb{C} \setminus \{0\}$ . If  $f \in \mathcal{A}_p$  and  $g \in \mathcal{A}_p$  satisfies the subordination

$$(2.4) \quad \begin{aligned} \Psi(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_3) < \alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} \\ + \delta_1 z q'(z) + \delta_2 \frac{z q'(z)}{q(z)} + \delta_3 \frac{z q'(z)}{(q(z))^2}, \end{aligned}$$

then

$$(2.5) \quad \left( \frac{(f * g)'(z)}{p z^{p-1}} \right)^\mu < q(z)$$

and  $q$  is the best dominant.

**Proof.** Define the function  $\psi$  by

$$(2.6) \quad \psi(z) := \left( \frac{(f * g)'(z)}{p z^{p-1}} \right)^\mu$$

so that, by a straightforward computation, we have

$$(2.7) \quad \frac{z \psi'(z)}{\psi(z)} = \mu \left[ \frac{z(f * g)''(z)}{(f * g)'(z)} - p - 1 \right].$$

Using (2.6) and (2.7) in (2.3) and (2.4), we obtain

$$\begin{aligned} \alpha + \xi \psi(z) + \delta(\psi(z))^2 + \frac{\gamma_1}{\psi(z)} + \delta_1 z \psi'(z) + \delta_2 \frac{z \psi'(z)}{\psi(z)} + \delta_3 \frac{z \psi'(z)}{(\psi(z))^2} \\ < \alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} + \delta_1 z q'(z) + \delta_2 \frac{z q'(z)}{q(z)} + \delta_3 \frac{z q'(z)}{(q(z))^2}. \end{aligned}$$

By setting

$$\theta(\omega) := \alpha + \xi \omega + \delta \omega^2 + \frac{\gamma_1}{\omega} \quad \text{and} \quad \phi(\omega) := \delta_1 + \frac{\delta_2}{\omega} + \frac{\delta_3}{\omega^2},$$

we obtain

$$\theta(\psi(z)) + z \psi'(z) \phi(\psi(z)) < \theta(q(z)) + z q'(z) \phi(q(z)).$$

It can be easily observed that  $\theta$  and  $\phi$  are analytic in  $\mathbb{C} \setminus \{0\}$  and that

$$\phi(\omega) \neq 0 \quad (\omega \in \mathbb{C} \setminus \{0\}).$$

Also, by letting

$$(2.8) \quad Q(z) = z q'(z) \phi(q(z)) = \delta_1 z q'(z) + \delta_2 \frac{z q'(z)}{q(z)} + \delta_3 \frac{z q'(z)}{(q(z))^2}$$

and

$$(2.9) \quad \begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= \alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} \\ &\quad + \delta_1 z q'(z) + \delta_2 \frac{z q'(z)}{q(z)} + \delta_3 \frac{z q'(z)}{(q(z))^2}, \end{aligned}$$

we find from (2.2) that  $Q$  is starlike univalent in  $\Delta$  and that

$$\Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left\{ 1 + \left( \frac{\xi q^2(z) + 2\delta q^3(z) - \gamma_1}{\delta_1 q^2(z) + \delta_2 q(z) + \delta_3} \right) - \frac{zq'(z)}{q(z)} \left\{ \frac{\delta_2 q(z) + 2\delta_3}{\delta_1 q^2(z) + \delta_2 q(z) + \delta_3} \right\} + \frac{zq''(z)}{q'(z)} \right\} > 0,$$

( $z \in \Delta$ ;  $\alpha, \delta, \xi, \gamma_1, \delta_1, \delta_2, \delta_3 \in \mathbb{C}$ ) by the hypothesis (2.1) and (2.2). The assertion (2.5) now follows by an application of Theorem A.  $\square$

**Remark 1.** For the choices  $p = 1$ ,  $g(z) = \frac{z}{1-z}$ ,  $\xi = -\xi$ ,  $\delta = \delta_2 = \delta_3 = \gamma_1 = 0$ ,  $q(z) = 1 + \frac{\lambda z}{k} \int_0^1 \frac{t^\xi}{1 + \frac{z}{k}t} dt$  and  $\delta_1 = -1$ , in Theorem 1, we get the corresponding result obtained by Singh [23, Theorem 1 (ii), p. 571]

**Remark 2.** For the choices  $p = 1$ ,  $g(z) = \frac{z}{1-z}$ ,  $\alpha = -\alpha$ ,  $\delta = \delta_2 = \delta_3 = \gamma_1 = 0$ ,  $\mu = -1$   $q(z) = 1 + \frac{\lambda}{1+\xi}z$  and  $\delta_1 = 1$ , in Theorem 1, we get the corresponding result obtained by Singh [23, Theorem 2 (ii), p. 572]

For a special case when  $p = 1$ ,  $g(z) = \frac{z}{1-z}$ ,  $q(z) = \frac{1}{(1-z)^{2b}}$  ( $b \in \mathbb{C} \setminus \{0\}$ ),  $\delta = \xi = \gamma_1 = \delta_1 = \delta_3 = 0$ ,  $\mu = \alpha = 1$  and  $\delta_2 = \frac{1}{b}$ , Theorem 1 reduces to the following known result obtained by Srivastava and Lashin [24].

**Corollary 1.** *Let  $b$  be a non zero complex number. If  $f \in \mathcal{A}$  satisfies*

$$1 + \frac{1}{b} \left[ \frac{zf''(z)}{f'(z)} \right] \prec \frac{1+z}{1-z},$$

then

$$f'(z) \prec \frac{1}{(1-z)^{2b}}$$

and  $\frac{1}{(1-z)^{2b}}$  is the best dominant.

**Theorem 2.** *Let  $q$  be analytic with  $q(0) = 1$  and univalent in  $\Delta$  such that  $q(z) \neq 0$ . Let  $z \in \Delta$ ,  $\alpha, \delta, \xi, \gamma_1, \delta_1, \delta_2, \delta_3 \in \mathbb{C}$  and suppose at least one of  $\delta_1, \delta_2, \delta_3$  is non-zero. Let  $q$  satisfies (2.1) and (2.2). Let*

$$\begin{aligned} \Psi_1(f, g, \mu, \xi, \beta, \delta, \gamma_1, \delta_1, \delta_3) &:= \alpha + \xi \left[ \frac{(f * g)(z)}{z^p} \right]^\mu \\ &+ \delta \left[ \frac{(f * g)(z)}{z^p} \right]^{2\mu} + \gamma_1 \left[ \frac{z^p}{(f * g)(z)} \right]^\mu \\ &+ \mu \left( \frac{z(f * g)'(z)}{(f * g)(z)} - p \right) \left[ \delta_1 \left( \frac{(f * g)(z)}{z^p} \right)^\mu + \delta_2 \right] \\ &+ \mu \delta_3 \left[ \frac{z(f * g)'(z)}{(f * g)(z)} - p \right] \left( \frac{(f * g)(z)}{z^p} \right)^\mu. \end{aligned} \tag{2.10}$$

If  $f \in \mathcal{A}_p$  and  $g \in \mathcal{A}_p$  satisfies the subordination

$$(2.11) \quad \begin{aligned} \Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_3) < \alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} \\ + \delta_1 z q'(z) + \delta_2 \frac{z q'(z)}{q(z)} + \delta_3 \frac{z q'(z)}{(q(z))^2} \end{aligned}$$

for some  $\mu \in \mathbb{C} \setminus \{0\}$ , then

$$(2.12) \quad \left( \frac{(f * g)(z)}{z^p} \right)^\mu < q(z)$$

and  $q$  is the best dominant.

**Proof.** Let the function  $\psi$  be defined by

$$(2.13) \quad \psi(z) := \left( \frac{(f * g)(z)}{z^p} \right)^\mu.$$

Evidently,

$$(2.14) \quad \frac{z\psi'(z)}{\psi(z)} = \mu \left[ \frac{z(f * g)'(z)}{(f * g)(z)} - p \right].$$

In view of (2.13) and (2.14), it follows from (2.10) and (2.11),

$$\begin{aligned} \alpha + \xi \psi(z) + \delta(\psi(z))^2 + \frac{\gamma_1}{\psi(z)} + \delta_1 z \psi'(z) + \delta_2 \frac{z \psi'(z)}{\psi(z)} + \delta_3 \frac{z \psi'(z)}{(\psi(z))^2} \\ < \alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} + \delta_1 z q'(z) + \delta_2 \frac{z q'(z)}{q(z)} + \delta_3 \frac{z q'(z)}{(q(z))^2}. \end{aligned}$$

Letting  $\theta$  and  $\phi$  as defined in Theorem 1 and following the steps of Theorem 1, the assertions (2.1) and (2.2), the result follows by an application of Theorem A.  $\square$

**Remark 3.** For the choices

$$p = 1, \quad g(z) = \phi(a, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^n, \quad \gamma_1 = \delta_1 = \delta_3 = 0,$$

Theorem 1 coincides with the result obtained by Shanmugam *et al.* [20].

**Remark 4.** For the choices

$$p = 1, \quad g(z) = z + \sum_{n=2}^{\infty} \left( \frac{n + \lambda}{1 + \lambda} \right)^k z^n \quad (\lambda \geq 0; k \in \mathbb{Z}), \quad \gamma_1 = \delta_1 = \delta_3 = 0,$$

Theorem 1 reduces to the result obtained by Shanmugam *et al.* [18].

**Remark 5.** For the choices

$$p = 1, \quad g(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} (\alpha_2)_{n-1} \cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} (\beta_2)_{n-1} \cdots (\beta_s)_{n-1} (1)_{n-1}} z^n, \quad \gamma_1 = \delta_1 = \delta_3 = 0,$$



Theorem 1 coincides with the corresponding result obtained by Murugusundaramoorthy and Magesh [10].

For a special case  $p = 1$ ,  $q(z) = e^{\mu Az}$ , with  $|\mu A| < \pi$ , Theorem 1 readily yields the following.

**Corollary 2.** *Assume that (2.1) holds. If  $f \in \mathcal{A}$ , and*

$$\begin{aligned} \Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3) < \alpha + \xi e^{\mu Az} + \delta e^{2\mu Az} + \gamma_1 e^{-\mu Az} \\ + \delta_1 z A \mu e^{\mu Az} + \delta_2 A \mu z + \delta_3 A \mu z e^{\mu Az} \end{aligned}$$

( $z \in \Delta$ ;  $\alpha, \delta, \xi, \gamma_1, \delta_1, \delta_2, \delta_3 \in \mathbb{C}$ ;  $\delta_2 \neq 0$ ) where  $\Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_3)$  is as defined in (2.10), then

$$\left( \frac{(f * g)(z)}{z} \right)^\mu < e^{\mu Az} \quad (z \in \Delta; z \neq 0; \mu \in \mathbb{C}, \mu \neq 0)$$

and  $e^{\mu Az}$  is the best dominant.

**Remark 6.** Taking  $p = 1$ ,  $g(z) = \frac{z}{1-z}$ ,  $\delta = \xi = \gamma_1 = \delta_1 = \delta_3 = 0$ ,  $\alpha = 1$ ,  $\delta_2 = \frac{1}{\mu}$  in Corollary 2, we get the result obtained by Obradović and Owa [11].

For a special case when  $p = 1$ ,  $g(z) = \frac{z}{1-z}$ ,  $q(z) = \frac{1}{(1-z)^{2b}}$  ( $b \in \mathbb{C} \setminus \{0\}$ ),  $\delta = \xi = \gamma_1 = \delta_1 = \delta_3 = 0$ ,  $\mu = \alpha = 1$  and  $\delta_2 = \frac{1}{b}$ , Theorem 1 reduces at once to the following known result obtained by Srivastava and Lashin [24].

**Corollary 3.** *Let  $b$  be a non-zero complex number. If  $f \in \mathcal{A}$  satisfies*

$$1 + \frac{1}{b} \left[ \frac{z f'(z)}{f(z)} - 1 \right] < \frac{1+z}{1-z},$$

then

$$\frac{f(z)}{z} < \frac{1}{(1-z)^{2b}}$$

and  $\frac{1}{(1-z)^{2b}}$  is the best dominant.

If we put  $p = 1$ ,  $g(z) = \frac{z}{1-z}$ ,  $q(z) = (1+Bz)^{\frac{\mu(A-B)}{B}}$ ,  $\delta = \xi = \gamma_1 = \delta_1 = \delta_3 = 0$ ,  $\alpha = 1$ ,  $\delta_2 = \frac{1}{\mu}$  in Theorem 1, we get the following known result obtained by Obradović and Owa [11].

**Corollary 4.** *Let  $-1 \leq B < A \leq 1$ . Let  $\mu, A, B$  satisfy the relation either  $\left| \frac{\mu(A-B)}{B} - 1 \right| \leq 1$  or  $\left| \frac{\mu(A-B)}{B} + 1 \right| \leq 1$ . If  $f \in \mathcal{A}$  satisfies*

$$\frac{z f'(z)}{f(z)} < \frac{1+Az}{1+Bz},$$

then

$$\left( \frac{f(z)}{z} \right)^\mu < (1+Bz)^{\frac{\mu(A-B)}{B}} \quad (z \in \Delta; z \neq 0; \mu \in \mathbb{C}; \mu \neq 0)$$

and  $(1 + Bz)^{\frac{\mu(A-B)}{B}}$  is the best dominant.

For the next four theorems, we assume that  $Q(z) = zq'(z)\phi(q(z))$ , where  $\phi$  is analytic in  $\mathbb{C} \setminus \{0\}$ . Next, by appealing to Theorem B we prove two superordination results in Theorem 3 and Theorem 4.

**Theorem 3.** *Let  $q$  be analytic and univalent in  $\Delta$  such that  $q(z) \neq 0$ . Let  $z \in \Delta$ ,  $\delta, \xi, \gamma_1, \delta_1, \delta_2, \delta_3 \in \mathbb{C}$  and  $\mu \in \mathbb{C} \setminus \{0\}$ . Suppose that  $q$  satisfies (2.2) and*

$$(2.15) \quad \Re \left[ \frac{2\delta q^3(z) + \xi q^2(z) - \gamma_1}{\delta_1 q^2(z) + \delta_2 q(z) + \delta_3} \right] > 0.$$

If  $f \in \mathcal{A}_p$ ,  $\Psi(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$  defined by (2.3) is univalent in  $\Delta$ , and  $\left(\frac{(f*g)'(z)}{pz^{p-1}}\right)^\mu \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$  satisfy the subordination

$$\begin{aligned} \alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} + \delta_1 zq'(z) + \delta_2 \frac{zq'(z)}{q(z)} + \delta_3 \frac{zq'(z)}{(q(z))^2} \\ \prec \Psi(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3), \end{aligned}$$

then

$$(2.16) \quad q(z) \prec \left(\frac{(f*g)'(z)}{pz^{p-1}}\right)^\mu$$

and  $q$  is the best subordinant.

**Proof.** Defining  $\psi$  by (2.6) and following the steps of the proof of Theorem 1, we have

$$\begin{aligned} \alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} + \delta_1 zq'(z) + \delta_2 \frac{zq'(z)}{q(z)} + \delta_3 \frac{zq'(z)}{(q(z))^2} \\ \prec \alpha + \xi \psi(z) + \delta(\psi(z))^2 + \frac{\gamma_1}{\psi(z)} + \delta_1 z\psi'(z) + \delta_2 \frac{z\psi'(z)}{\psi(z)} + \delta_3 \frac{z\psi'(z)}{(\psi(z))^2}. \end{aligned}$$

Setting

$$\vartheta(w) := \alpha + \xi w + \delta w^2 + \frac{\gamma_1}{w} \quad \text{and} \quad \varphi(w) := \delta_1 + \frac{\delta_2}{w} + \frac{\delta_3}{w^2},$$

we observe that  $\vartheta$  and  $\varphi$  are analytic in  $\mathbb{C} \setminus \{0\}$  and that

$$\varphi(w) \neq 0 \quad \text{in} \quad \mathbb{C} \setminus \{0\}.$$

It follows that

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(\psi(z)) + z\psi'(z)\varphi(\psi(z)).$$

In view of the given conditions (2.15) and (2.2) and since  $q$  is univalent, it is routine to show that (1) and (2) of Theorem B are satisfied. The assertion (2.16) follows by an application of Theorem B.  $\square$

**Theorem 4.** Let  $q$  be analytic and univalent in  $\Delta$  such that  $q(z) \neq 0$ . Let  $z \in \Delta$ ,  $\delta, \xi, \gamma_1, \delta_1, \delta_2, \delta_3 \in \mathbb{C}$  and  $\mu \in \mathbb{C} \setminus \{0\}$ . Suppose that  $q$  satisfies (2.15). If  $f \in \mathcal{A}_p$ ,

$$\left( \frac{(f * g)(z)}{z^p} \right)^\mu \in \mathcal{H}[q(0), 1] \cap \mathcal{Q},$$

and  $\Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$  defined by (2.10) is univalent in  $\Delta$ , then

$$\begin{aligned} \alpha + \xi q(z) + \delta(q(z))^2 + \frac{\gamma_1}{q(z)} + \delta_1 z q'(z) + \delta_2 \frac{z q'(z)}{q(z)} + \delta_3 \frac{z q'(z)}{(q(z))^2} \\ \prec \Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3) \end{aligned}$$

implies

$$(2.17) \quad q(z) \prec \left( \frac{(f * g)(z)}{z^p} \right)^\mu$$

and  $q$  is the best subordinant.

**Proof.** Let the function  $\psi$  be defined by (2.13). By setting

$$\vartheta(w) := \alpha + \xi w + \delta w^2 + \frac{\gamma_1}{w} \quad \text{and} \quad \varphi(w) := \delta_1 + \frac{\delta_2}{w} + \frac{\delta_3}{w^2},$$

it is easily observed that the functions  $\vartheta$  and  $\varphi$  are analytic in  $\mathbb{C} \setminus \{0\}$  and that

$$\varphi(w) \neq 0, \quad (w \in \mathbb{C} \setminus \{0\}).$$

The assertion (2.17) follows by an application of Theorem B.  $\square$

Combining the results of differential subordination in Theorem 1 and superordination in Theorem 3, we state the following ‘‘sandwich’’ result.

**Theorem 5.** Let  $q_1$  and  $q_2$  be univalent in  $\Delta$  such that  $q_1(z) \neq 0$  and  $q_2(z) \neq 0$ . Suppose  $q_1$  and  $q_2$  satisfy respectively, (2.2), (2.15) and (2.1), (2.2). Let  $z \in \Delta$ ,  $\delta, \xi, \gamma_1, \delta_1, \delta_2, \delta_3 \in \mathbb{C}$  and  $\mu \in \mathbb{C} \setminus \{0\}$ . If  $f \in \mathcal{A}_p$ ,  $\left( \frac{(f * g)'(z)}{p z^{p-1}} \right)^\mu \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$  and  $\Psi(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$  defined by (2.3) is univalent in  $\Delta$ , then

$$\begin{aligned} \alpha + \xi q_1(z) + \delta(q_1(z))^2 + \frac{\gamma_1}{q_1(z)} + \delta_1 z q_1'(z) + \delta_2 \frac{z q_1'(z)}{q_1(z)} + \delta_3 \frac{z q_1'(z)}{(q_1(z))^2} \\ \prec \Psi(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3) \\ \prec \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \frac{\gamma_1}{q_2(z)} + \delta_1 z q_2'(z) + \delta_2 \frac{z q_2'(z)}{q_2(z)} + \delta_3 \frac{z q_2'(z)}{(q_2(z))^2} \end{aligned}$$

implies

$$q_1(z) \prec \left( \frac{(f * g)'(z)}{p z^{p-1}} \right)^\mu \prec q_2(z)$$

and  $q_1$  and  $q_2$  are respectively the best subordinant and best dominant.

Finally, combining Theorem 2 and Theorem 4 we obtain the following sandwich theorem.

**Theorem 6.** *Let  $q_1$  and  $q_2$  be univalent in  $\Delta$  such that  $q_1(z) \neq 0$  and  $q_2(z) \neq 0$ . Suppose  $q_1$  satisfy (2.2) and (2.15), and  $q_2$  satisfy (2.1) and (2.2). Let  $z \in \Delta$ ,  $\delta, \xi, \gamma_1, \delta_1, \delta_2, \delta_3 \in \mathbb{C}$  and  $\mu \in \mathbb{C} \setminus \{0\}$ . If  $f \in \mathcal{A}_p$ ,  $\left(\frac{(f * g)(z)}{z^p}\right)^\mu \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$  and  $\Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3)$  defined by (2.10) is univalent in  $\Delta$ , then*

$$\begin{aligned} & \alpha + \xi q_1(z) + \delta(q_1(z))^2 + \frac{\gamma_1}{q_1(z)} + \delta_1 z q_1'(z) + \delta_2 \frac{z q_1'(z)}{q_1(z)} + \delta_3 \frac{z q_1'(z)}{(q_1(z))^2} \\ & \prec \Psi_1(f, g, \mu, \xi, \delta, \gamma_1, \delta_1, \delta_2, \delta_3) \\ & \prec \alpha + \xi q_2(z) + \delta(q_2(z))^2 + \frac{\gamma_1}{q_2(z)} + \delta_1 z q_2'(z) + \delta_2 \frac{z q_2'(z)}{q_2(z)} + \delta_3 \frac{z q_2'(z)}{(q_2(z))^2} \end{aligned}$$

implies

$$q_1(z) \prec \left(\frac{(f * g)(z)}{z^p}\right)^\mu \prec q_2(z)$$

and  $q_1$  and  $q_2$  are respectively the best subdominant and best dominant.

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#### REFERENCES

- [1] Ali, R. M., Ravichandran, V., Khan, M. Hussain and Subramanian, K. G., *Differential sandwich theorems for certain analytic functions*, Far East J. Math. Sci. **15** (2004), no. 1, 87–94.
- [2] Bulboacă, T., *A class of superordination-preserving integral operators*, Indag. Math. (N. S.) **13** (2002), no. 3, 301–311.
- [3] Bulboacă, T., *Classes of first-order differential subordinations*, Demonstratio Math. **35** (2002), no. 2, 287–292.
- [4] Carlson, B. C., Shaffer, D. B., *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15** (1984), 737–745.
- [5] Cho, N. E., Kim, T. H., *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc. **40** (2003), no. 3, 399–410.
- [6] Cho, N. E., Srivastava, H. M., *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling **37** (2003), no. 1–2, 39–49.
- [7] Dziok, J., Srivastava, H. M., *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. **103** (1999), no. 1, 1–13.
- [8] Miller, S. S., Mocanu, P. T., *Differential Subordinations. Theory and Applications*, Marcel Dekker, New York, 2000.
- [9] Miller, S. S., Mocanu, P. T., *Subordinants of differential subordinations*, Complex Var. Theory Appl. **48** (2003), no. 10, 815–826.
- [10] Murugusundaramoorthy, G., Magesh, N., *Differential subordinations and superordinations for analytic functions defined by the Dziok–Srivastava linear operator*, J. Inequal. Pure Appl. Math. **7** (2006), no. 4, Art. 152, 9 pp. (electronic).
- [11] Obradović, M., Owa, S., *On certain properties for some classes of starlike functions*, J. Math. Anal. Appl. **145** (1990), no. 2, 357–364.

- [12] Patel, J., Cho, N. E., *Some classes of analytic functions involving Noor integral operator*, J. Math. Anal. Appl. **312** (2005), no. 2, 564–575.
- [13] Ravichandran, V., Darus, M., Khan, Hussain M. and Subramanian, K. G., *Differential subordination associated with certain linear operators defined by multivalent functions*, Acta Math. Vietnam. **30** (2005), no. 2, 113–121.
- [14] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.
- [15] Sălăgean, G. Ş., *Subclasses of univalent functions*, Complex analysis — fifth Romanian–Finnish seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., 1013, Springer, Berlin, 1983, 362–372.
- [16] Shanmugam, T. N., *Convolution and differential subordination*, Internat. J. Math. Math. Sci. **12** (1989), no. 2, 333–340.
- [17] Shanmugam, T. N., Ravichandran, V. and Sivasubramanian, S., *Differential sandwich theorems for some subclasses of analytic functions*, Aust. J. Math. Anal. Appl. **3** (2006), no. 1, Art. 8, 11 pp. (electronic).
- [18] Shanmugam, T. N., Sivasubramanian, S. and Srivastava, H. M., *Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations*, Integral Transforms Spec. Functions **17** (2006), no. 12, 889–899.
- [19] Shanmugam, T. N., Sivasubramanian, S. and Darus, M., *On certain subclasses of functions involving a linear operator*, Far East J. Math. Sci. **23** (2006), no. 3, 329–339.
- [20] Shanmugam, T. N., Sivasubramanian, S. and Owa, S., *On sandwich theorems for some subclasses of analytic functions involving a linear operator*, to appear in Math. Inequal. Appl.
- [21] Shanmugam, T. N., Sivasubramanian, S. and Silverman, H., *On sandwich theorems for some classes of analytic functions*, Int. J. Math. Math. Sci. (2006) Art. ID 29864, 13 pp.
- [22] Shenan, Gamal M., Tariq, Salim O. and Mousa, Marouf S., *A certain class of multivalent prestarlike functions involving the Srivastava–Saigo–Owa fractional integral operator*, Kyungpook Math. J. **44** (2004), no. 3, 353–362.
- [23] Singh, V., *On some criteria for univalence and starlikeness*, Indian J. Pure Appl. Math. **34** (2003), no. 4, 569–577.
- [24] Srivastava, H. M., Lashin, A. Y., *Some applications of the Briot–Bouquet differential subordination*, JIPAM. J. Inequal. Pure Appl. Math. **6** (2005), no. 2, Art. 41, 7 pp. (electronic).

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