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Differential sandwich theorems for analytic functions defined by some linear operators

Abstract. In this investigation, we obtain some applications of first order differential subordination and superordination results involving Dziok-Srivastava operator and other linear operators for certain normalized analytic functions. Some of our results improve previous results.

1. Introduction. Let $H(U)$ be the class of analytic functions in the unit disk $U = \{z \in C : |z| < 1\}$ and let $H[a, k]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} \dots \ (a \in C). \tag{1.1}$$

For simplicity, let $H[a] = H[a, 1]$. Also, let A be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + a_2 z^2 + \dots \ . \tag{1.2}$$

If $f,\ g \in H(U)$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the

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following equivalence, (cf., e.g., [4], [12]; see also [13]):

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $p, h \in H(U)$ and let $\varphi(r, s, t; z) : C^3 \times U \to C$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is a solution of the differential superordination (1.3). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a subordinant if $q(z) \prec p(z)$ for all p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants of (1.3) is called the best subordinant. Recently Miller and Mocanu [14] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

Using the results of Miller and Mocanu [14], Bulboača [3] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [5]. Ali et al. [1], have used the results of Bulboača [3] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U. Also, Tuneski [18] obtained a sufficient condition for starlikeness of f in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Recently, Shanmugam et al. [16] obtained sufficient conditions for the normalized analytic function f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

They [16] also obtained results for functions defined by using Carlson–Shaffer operator.

For complex numbers $\alpha_1, \alpha_2, \dots, \alpha_l$ and $\beta_1, \beta_2, \dots, \beta_s$ ($\beta_j \notin Z_0^- = \{0, -1, -2, \dots\}$; $j = 1, 2, \dots, s$), we define the generalized hypergeometric function ${}_lF_s(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_s; z)$ by (see, for example, [17]) by the following infinite series:

$${}_lF_s(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_l)_k}{(\beta_1)_k \dots (\beta_s)_k (1)_k} z^k$$

$$(l \leq s + 1;\ s, l \in N_0 = N \cup \{0\};\ z \in U), \tag{1.5}$$

where

$$(d)_k = \begin{cases} 1 & (k = 0;\ d \in C \setminus \{0\}) \\ d(d+1)\dots(d+k-1) & (k \in N;\ d \in C). \end{cases}$$

Dziok and Srivastava [9] (see also [10]) considered a linear operator $H_{l,s}(\alpha_1, \dots, \alpha_q;\ \beta_1, \dots, \beta_s) : A \to A$, defined by the following Hadamard product:

$$H_{l,s}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_s)f(z) = [z\ {}_lF_s(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_s; z)] * f(z),$$

$$(l \leq s+1;\ s, l \in N_0;\ z \in U). \tag{1.6}$$

We observe that for a function f of the form (1.2), we have

$$\begin{aligned} &H_{l,s}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_s)f(z) \\ &\qquad = z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1} \dots (\alpha_l)_{k-1}}{(\beta_1)_{k-1} \dots (\beta_s)_{k-1}(1)_{k-1}} a_k z^k. \end{aligned} \tag{1.7}$$

If, for convenience, we write

$$H_{l,s}(\alpha_1) = H_{l,s}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_s), \tag{1.8}$$

then one can easily verify from the definition (1.7) that

$$z(H_{l,s}(\alpha_1)f(z))' = \alpha_1 H_{l,s}(\alpha_1+1)f(z) - (\alpha_1 - 1)H_{l,s}(\alpha_1)f(z) \tag{1.9}$$

$(f(z) \in A)$.

It should be remarked that the linear operator $H_{l,s}(\alpha_1)f(z)$ is a generalization of many other linear operators considered earlier. In particular, for $f \in A$, we have:

(i) $H_{2,1}(a, 1; c)f(z) = L(a, c)f(z)$ $(a > 0;\ c > 0)$, where $L(a, c)$ is the Carlson–Shaffer operator (see [6]);

(ii) $H_{2,1}(\lambda + 1, c; a)f(z) = I^{\lambda}(a, c)f(z)$ $(a, c \in R \setminus Z_0^-;\ \lambda > -1)$, where $I^{\lambda}(a, c)f(z)$ is the Cho–Kwon–Srivastava operator (see [7]);

(iii) $H_{2,1}(\mu, 1; \lambda + 1)f(z) = I_{\lambda,\mu}f(z)$ $(\lambda > -1;\ \mu > 0)$, where $I_{\lambda,\mu}f(z)$ is the Choi–Saigo–Srivastava operator (see [8]);

(iv) $H_{2,1}(\mu + 1, 1; \mu + 2)f(z) = F_{\mu}(f)(z) = \frac{\mu+1}{z^{\mu}} \int_0^z t^{\mu-1} f(t)dt$ $(\mu > -1)$ where $F_{\mu}(f)(z)$ is the Libera operator (see [11] and [2]);

(v) $H_{2,1}(\delta + 1, 1; 1)f(z) = D^{\delta}f(z)$ $(\delta > -1)$, where $D^{\delta}f(z)$ is the δ-Ruscheweyh derivative of $f(z)$ (see [15]).

In this paper, we obtain sufficient conditions for the normalized analytic function f defined by using Dziok–Srivastava operator to satisfy:

$$q_1(z) \prec \frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{H_{l,s}(\alpha_1+1)f(z)}{\{H_{l,s}(\alpha_1)f(z)\}^2} \prec q_2(z)$$

and q_1 and q_2 are given univalent functions in U.

2. Definitions and preliminaries. In order to prove our results, we shall make use of the following known results.

Definition 1 ([14]). Denote by Q, the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \xi \in \partial U : \lim_{z \to \xi} f(z) = \infty \right\},$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(f)$.

Lemma 1 ([14]). *Let q be univalent in the unit disk U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set*

$$\psi(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z). \tag{2.1}$$

Suppose that

(i) $\psi(z)$ *is starlike univalent in* U,

(ii) $\operatorname{Re}\left\{ \frac{zh'(z)}{\psi(z)} \right\} > 0$ *for* $z \in U$.

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.2}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 1, Shanmugam et al. [16] obtained the following lemma.

Lemma 2 ([16]). *Let q be univalent in U with $q(0) = 1$. Let $\alpha \in C$; $\gamma \in C^* = C \setminus \{0\}$, further assume that*

$$\operatorname{Re}\left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max\{0, -\operatorname{Re}(\alpha/\gamma)\}.$$

If p is analytic in U, and

$$\alpha p(z) + \gamma zp'(z) \prec \alpha q(z) + \gamma zq'(z),$$

then $p \prec q$ and q is the best dominant.

Lemma 3 ([3]). *Let q be convex univalent in U and ϑ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\operatorname{Re}\{\vartheta'(q(z))/\phi(q(z))\} > 0$ *for* $z \in U$,

(ii) $\psi(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* U.

If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)), \tag{2.3}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Taking $\theta(w) = \alpha w$ and $\varphi(w) = \gamma$ in Lemma 3, Shanmugam et al. [16] obtained the following lemma.

Lemma 4 ([16]). *Let q be convex univalent in U, $q(0) = 1$. Let $\alpha \in C$, $\gamma \in C^*$ and $\operatorname{Re}\{\alpha/\gamma\} > 0$. If $p \in H[q(0), 1] \cap Q$, $\alpha p(z) + \gamma z p'(z)$ is univalent in U and*

$$\alpha q(z) + \gamma z q'(z) \prec \alpha p(z) + \gamma z p'(z),$$

then $q \prec p$ and q is the best subordinant.

3. Applications to Dziok–Srivastava operator and sandwich theorems.

Theorem 1. *Let q be convex univalent in U with $q(0) = 1$, $\gamma \in C^*$. Further, assume that*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0, -\operatorname{Re}(1/\gamma)\}. \tag{3.1}$$

If $f \in A$, $H_{l,s}(\alpha_1 + 1)f(z) \neq 0$ for $0 < |z| < 1$, and

$$\begin{aligned} &\gamma\alpha_1 + (1+\gamma)\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} \\ &\quad - \gamma(1+\alpha_1)\frac{H_{l,s}(\alpha_1+2)f(z)H_{l,s}(\alpha_1)f(z)}{\{H_{l,s}(\alpha_1+1)f(z)\}^2} \\ &\qquad \prec q(z) + \gamma z q'(z), \end{aligned} \tag{3.2}$$

then

$$\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define a function p by

$$p(z) = \frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} \quad (z \in U). \tag{3.3}$$

Then the function p is analytic in U and $p(0) = 1$. Therefore, differentiating (3.3) logarithmically with respect to z and using the identity (1.9) in the resulting equation, we have

$$\gamma\alpha_1 + (1+\gamma)\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} - \gamma(1+\alpha_1)\frac{H_{l,s}(\alpha_1+2)f(z)H_{l,s}(\alpha_1)f(z)}{\{H_{l,s}(\alpha_1+1)f(z)\}^2} = p(z) + \gamma z p'(z),$$

that is,

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z)$$

and therefore, the theorem follows by applying Lemma 2. □

Putting $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \le B < A \le 1)$ in Theorem 1, we have the following corollary.

Corollary 1. *If $f(z) \in A$ and $\gamma \in C^*$ satisfy*

$$\gamma\alpha_1 + (1+\gamma)\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} - \gamma(1+\alpha_1)\frac{H_{l,s}(\alpha_1+2)f(z)H_{l,s}(\alpha_1)f(z)}{\{H_{l,s}(\alpha_1+1)f(z)\}^2}$$
$$\prec \gamma\frac{(A-B)z}{(1+Bz)^2} + \frac{1+Az}{1+Bz},$$

then

$$\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} \prec \frac{1+Az}{1+Bz}.$$

Putting $A = 1$, $B = -1$ and $q(z) = \frac{1+z}{1-z}$ in Corollary 1, we have

Corollary 2. *If $f(z) \in A$ and $\gamma \in C^*$ satisfy*

$$\gamma\alpha_1 + (1+\gamma)\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} - \gamma(1+\alpha_1)\frac{H_{l,s}(\alpha_1+2)f(z)H_{l,s}(\alpha_1)f(z)}{\{H_{l,s}(\alpha_1+1)f(z)\}^2}$$
$$\prec \frac{2\gamma z}{(1-z)^2} + \frac{1+z}{1-z},$$

then

$$\text{Re}\left\{\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)}\right\} > 0.$$

Taking $\alpha_1 = a > 0$, $\beta_1 = c > 0$, $\alpha_j = 1$ $(j = 2,\ldots,s+1)$ and $\beta_j = 1$ $(j = 2,\ldots,s)$, in Theorem 1, we have the following corollary which improves the result of Shanmugam et al. [16, Theorem 4.1].

Corollary 3. *Let q be convex univalent in U with $q(0) = 1$, $\gamma \in C^*$. Further, assume that* (3.1) *holds. If $f \in A$, and*

$$\gamma a + (1+\gamma)\frac{L(a,c)f(z)}{L(a+1,c)f(z)} - \gamma(1+a)\frac{L(a+2,c)f(z)L(a,c)f(z)}{\{L(a+1,c)f(z)\}^2}$$
$$\prec q(z) + \gamma z q'(z),$$

then

$$\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \prec q(z)$$

and q is the best dominant.

Taking $\alpha_1 = \lambda + 1$, $\alpha_2 = c$, $\beta_1 = a$ $(a, c \in R \setminus Z_o^-;\ \lambda > -1)$, $\alpha_j = 1$ $(j = 3,\ldots,s+1)$ and $\beta_j = 1$ $(j = 2,\ldots,s)$, in Theorem 1, we have

Corollary 4. *Let q be convex univalent in U with $q(0) = 1$, $\gamma \in C^*$. Further, assume that* (3.1) *holds. If $f \in A$, and*

$$\gamma(\lambda+1) + (1+\gamma)\frac{I^{\lambda}(a,c)f(z)}{I^{\lambda+1}(a,c)f(z)} - \gamma(\lambda+2)\frac{I^{\lambda+2}(a,c)f(z)I^{\lambda}(a,c)f(z)}{\{I^{\lambda+1}(a,c)f(z)\}^2}$$
$$\prec q(z) + \gamma z q'(z),$$

then

$$\frac{I^{\lambda}(a,c)f(z)}{I^{\lambda+1}(a,c)f(z)} \prec q(z)$$

and q is the best dominant.

Taking $\alpha_1 = \mu$, $\beta_1 = \lambda + 1$ $(\lambda > -1;\ \mu > 0)$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$ in Theorem 1, we have

Corollary 5. *Let q be convex univalent in U with $q(0) = 1$, $\gamma \in C^*$. Further, assume that* (3.1) *holds. If $f \in A$, and*

$$\gamma\mu + (1+\gamma)\frac{I_{\lambda,\mu}f(z)}{I_{\lambda,\mu+1}f(z)} - \gamma(\mu+1)\frac{I_{\lambda,\mu+2}f(z)I_{\lambda,\mu}f(z)}{\{I_{\lambda,\mu}f(z)\}^2} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{I_{\lambda,\mu}f(z)}{I_{\lambda,\mu+1}f(z)} \prec q(z)$$

and q is the best dominant.

Taking $\alpha_1 = \mu + 1$, $\beta_1 = \mu + 2$ $(\mu > -1)$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$ in Theorem 1, we have

Corollary 6. *Let q be convex univalent in U with $q(0) = 1$, $\gamma \in C^*$. Further, assume that* (3.1) *holds. If $f \in A$, and*

$$\gamma(1+\mu) + (1-\gamma\mu)\frac{F_\mu f(z)}{f(z)} - \gamma\frac{zf'(z)F_\mu f(z)}{\{f(z)\}^2} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{F_\mu f(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

Now, by appealing to Lemma 4 it can be easily prove the following theorem.

Theorem 2. *Let q be convex univalent in U. Let $\gamma \in C$ with $\operatorname{Re}\gamma > 0$. If $f \in A$, $\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} \in H[1,1] \cap Q$,*

$$\gamma\alpha_1 + (1+\gamma)\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} - \gamma(1+\alpha_1)\frac{H_{l,s}(\alpha_1+2)f(z)H_{l,s}(\alpha_1)f(z)}{\{H_{l,s}(\alpha_1+1)f(z)\}^2}$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec \gamma\alpha_1 + (1+\gamma)\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} - \gamma(1+\alpha_1)\frac{H_{l,s}(\alpha_1+2)f(z)H_{l,s}(\alpha_1)f(z)}{\{H_{l,s}(\alpha_1+1)f(z)\}^2},$$

then

$$q(z) \prec \frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)}$$

and q is the best subordinant.

Taking $\alpha_1 = a > 0$, $\beta_1 = c > 0$, $\alpha_j = 1$ $(j = 2, \dots, s+1)$ and $\beta_j = 1$ $(j = 2, \dots, s)$ in Theorem 2, we have the following corollary which improve the result of Shanmugam et al. [16, Theorem 4.2].

Corollary 7. *Let q be convex univalent in U. Let $\gamma \in C$ with $\operatorname{Re}\gamma > 0$. If $f \in A$, $\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \in H[1,1] \cap Q$,*

$$\gamma a + (1+\gamma)\frac{L(a,c)f(z)}{L(a+1,c)f(z)} - \gamma(1+a)\frac{L(a+2,c)f(z)L(a,c)f(z)}{\{L(a+1,c)f(z)\}^2}$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec \gamma a + (1+\gamma)\frac{L(a,c)f(z)}{L(a+1,c)f(z)} - \gamma(1+a)\frac{L(a+2,c)f(z)L(a,c)f(z)}{\{L(a+1,c)f(z)\}^2},$$

then

$$q(z) \prec \frac{L(a,c)f(z)}{L(a+1,c)f(z)}$$

and q is the best subordinant.

Taking $\alpha_1 = \lambda + 1, \alpha_2 = c$, $\beta_1 = a$ $(a, c \in R \setminus Z_o^-;\ \lambda > -1)$, $\alpha_j = 1$ $(j = 3, \dots, s+1)$ and $\beta_j = 1$ $(j = 2, \dots, s)$, in Theorem 2, we have

Corollary 8. *Let q be convex univalent in U. Let $\gamma \in C$ with $\operatorname{Re}\gamma > 0$. If $f \in A$, $\frac{I^{\lambda}(a,c)f(z)}{I^{\lambda+1}(a,c)f(z)} \in H[1,1] \cap Q$,*

$$\gamma(\lambda+1) + (1+\gamma)\frac{I^{\lambda}(a,c)f(z)}{I^{\lambda+1}(a,c)f(z)} - \gamma(\lambda+2)\frac{I^{\lambda+2}(a,c)f(z)I^{\lambda}(a,c)f(z)}{\{I^{\lambda+1}(a,c)f(z)\}^2}$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec \gamma(\lambda+1) + (1+\gamma)\frac{I^{\lambda}(a,c)f(z)}{I^{\lambda+1}(a,c)f(z)} - \gamma(\lambda+2)\frac{I^{\lambda+2}(a,c)f(z)I^{\lambda}(a,c)f(z)}{\{I^{\lambda+1}(a,c)f(z)\}^2},$$

then

$$q(z) \prec \frac{I^{\lambda}(a,c)f(z)}{I^{\lambda+1}(a,c)f(z)}$$

and q is the best subordinant.

Taking $\alpha_1 = \mu$, $\beta_1 = \lambda + 1$ $(\lambda > -1;\ \mu > 0)$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$, in Theorem 2, we have

Corollary 9. *Let q be convex univalent in U. Let $\gamma \in C$ with $\operatorname{Re}\gamma > 0$. If $f \in A$, $\frac{I_{\lambda,\mu}f(z)}{I_{\lambda,\mu+1}f(z)} \in H[1,1] \cap Q$,*

$$\gamma\mu + (1+\gamma)\frac{I_{\lambda,\mu}f(z)}{I_{\lambda,\mu+1}f(z)} - \gamma(\mu+1)\frac{I_{\lambda,\mu+2}f(z)I_{\lambda,\mu}f(z)}{\{I_{\lambda,\mu}f(z)\}^2}$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec \gamma\mu + (1+\gamma)\frac{I_{\lambda,\mu}f(z)}{I_{\lambda,\mu+1}f(z)} - \gamma(\mu+1)\frac{I_{\lambda,\mu+2}f(z)I_{\lambda,\mu}f(z)}{\{I_{\lambda,\mu}f(z)\}^2},$$

then

$$q(z) \prec \frac{I_{\lambda,\mu}f(z)}{I_{\lambda,\mu+1}f(z)}$$

and q is the best subordinant.

Taking $\alpha_1 = \mu + 1$, $\beta_1 = \mu + 2$ $(\mu > -1)$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$, in Theorem 2, we have

Corollary 10. *Let q be convex univalent in U. Let $\gamma \in C$ with $\operatorname{Re}\gamma > 0$. If $f \in A$, $\frac{F_\mu f(z)}{f(z)} \in H[1,1] \cap Q$,*

$$\gamma(1+\mu) + (1-\gamma\mu)\frac{F_\mu f(z)}{f(z)} - \gamma\frac{zf'(z)F_\mu f(z)}{\{f(z)\}^2}$$

is univalent in U, and

$$q(z) + \gamma z q'(z) \prec \gamma(1+\mu) + (1-\gamma\mu)\frac{F_\mu f(z)}{f(z)} - \gamma\frac{zf'(z)F_\mu f(z)}{\{f(z)\}^2},$$

then

$$q(z) \prec \frac{F_\mu f(z)}{f(z)}$$

and q is the best dominant.

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem.

Theorem 3. *Let q_1 be convex univalent in U, $\gamma \in C$ with $\operatorname{Re}\gamma > 0$, q_2 be univalent in U, $q_2(0) = 1$ and satisfies (3.1). If $f \in A$, $\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} \in H[1,1] \cap Q$,*

$$\gamma\alpha_1 + (1+\gamma)\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} - \gamma(1+\alpha_1)\frac{H_{l,s}(\alpha_1+2)f(z)H_{l,s}(\alpha_1)f(z)}{\{H_{l,s}(\alpha_1+1)f(z)\}^2}$$

is univalent in U, and

$$q_1(z) + \gamma z q_1'(z) \prec \gamma\alpha_1 + (1+\gamma)\frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} - \gamma(1+\alpha_1)\frac{H_{l,s}(\alpha_1+2)f(z)H_{l,s}(\alpha_1)f(z)}{\{H_{l,s}(\alpha_1+1)f(z)\}^2} \prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{H_{l,s}(\alpha_1)f(z)}{H_{l,s}(\alpha_1+1)f(z)} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

4. Remarks. Combining: (i) Corollary 3 and Corollary 7; (ii) Corollary 4 and Corollary 8; (iii) Corollary 5 and Corollary 9; (iv) Corollary 6 and Corollary 10, we obtain similar sandwich theorems for the corresponding operators.

Theorem 4. *Let q be convex univalent in U, $\gamma \in C^*$. Further, assume that* (3.1) *holds. If $f \in A$ satisfies*

$$[1+\gamma(\alpha_1-1)]\frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2} + \gamma(1+\alpha_1)\frac{zH_{l,s}(\alpha_1+2)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2} - 2\gamma\alpha_1\frac{z[H_{l,s}(\alpha_1+1)f(z)]^2}{[H_{l,s}(\alpha_1)f(z)]^3} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2} \quad (z \in U).$$

Then, simple computations show that

$$p(z) + \gamma z p'(z) = [1+\gamma(\alpha_1-1)]\frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2} + \gamma(1+\alpha_1)\frac{zH_{l,s}(\alpha_1+2)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2} - 2\gamma\alpha_1\frac{z[H_{l,s}(\alpha_1+1)f(z)]^2}{[H_{l,s}(\alpha_1)f(z)]^3}.$$

Applying Lemma 2, the theorem follows. □

Taking $\alpha_1 = a > 0$, $\beta_1 = c > 0$, $\alpha_j = 1$ $(j = 2, \dots, s+1)$ and $\beta_j = 1$ $(j = 2, \dots, s)$ in Theorem 4, we have the following corollary which improves the result of Shanmugam et al. [16, Theorem 4.4].

Corollary 11. *Let q be convex univalent in U, $\gamma \in C^*$. Further, assume that* (3.1) *holds. If $f \in A$ satisfies*

$$[1+\gamma(a-1)]\frac{zL(a+1,c)f(z)}{[L(a,c)f(z)]^2} + \gamma(1+a)\frac{zL(a+2,c)f(z)}{[L(a,c)f(z)]^2} - 2\gamma a\frac{z[L(a+1,c)f(z)]^2}{[L(a,c)f(z)]^3} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{zL(a+1,c)f(z)}{[L(a,c)f(z)]^2} \prec q(z)$$

and q is the best dominant.

Taking $\alpha_1 = \lambda + 1$, $\alpha_2 = c$, $\beta_1 = a$ $(a, c \in R \setminus Z^-;\ \lambda > -1)$, $\alpha_j = 1$ $(j = 3, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$, in Theorem 4, we have

Corollary 12. *Let q be convex univalent in U, $\gamma \in C^*$. Further, assume that* (3.1) *holds. If $f \in A$ satisfies*

$$[1+\gamma(\lambda-1)]\frac{zI^{\lambda+1}(a,c)f(z)}{[I^{\lambda}(a,c)f(z)]^2} + \gamma(\lambda+2)\frac{zI^{\lambda+2}(a,c)f(z)}{[I^{\lambda}(a,c)f(z)]^2} - 2\gamma(\lambda+1)\frac{z[I^{\lambda+1}(a,c)f(z)]^2}{\left\{I^{\lambda}(a,c)f(z)\right\}^3} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{zI^{\lambda+1}(a,c)f(z)}{[I^{\lambda}(a,c)f(z)]^2} \prec q(z)$$

and q is the best dominant.

Taking $\alpha_1 = \mu$, $\beta_1 = \lambda + 1$ $(\lambda > -1;\ \mu > 0)$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$, in Theorem 4, we have

Corollary 13. *Let q be convex univalent in U, $\gamma \in C^*$. Further, assume that* (3.1) *holds. If $f \in A$ satisfies*

$$[1+\gamma(\mu-1)]\frac{zI_{\lambda,\mu+1}f(z)}{[I_{\lambda,\mu}f(z)]^2} + \gamma(\mu+1)\frac{zI_{\lambda,\mu+2}f(z)}{[I_{\lambda,\mu}f(z)]^2} - 2\gamma\mu\frac{z[I_{\lambda,\mu+1}f(z)]^2}{\{I_{\lambda,\mu}f(z)\}^3} \prec q(z) + \gamma z q'(z),$$

then

$$\frac{zI_{\lambda,\mu+1}f(z)}{[I_{\lambda,\mu}f(z)]^2} \prec q(z)$$

and q is the best dominant.

Taking $\alpha_1 = \mu + 1$, $\beta_1 = \mu + 2$ $(\mu > -1)$, $\alpha_j = 1$ $(j = 2, \ldots, s + 1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$, in Theorem 4, we have

Corollary 14. *Let q be convex univalent in U, $\gamma \in C^*$. Further, assume that* (3.1) *holds. If $f \in A$ satisfies*

$$[1+\gamma(1+2\mu)]\frac{zf(z)}{[F_\mu f(z)]^2}+\gamma\frac{z^2f'(z)}{[F_\mu f(z)]^2}-2\gamma(\mu+1)\frac{z(f(z))^2}{[F_\mu f(z)]^3} \prec q(z)+\gamma zq'(z),$$

then

$$\frac{zf(z)}{[F_\mu f(z)]^2} \prec q(z)$$

and q is the best dominant.

Theorem 5. *Let q be convex univalent in U. Let $\gamma \in C$ with* $\operatorname{Re}\gamma > 0$. *If $f \in A$, $\frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2} \in H[1,1] \cap Q$,*

$$[1+\gamma(\alpha_1-1)]\frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2}+\gamma(1+\alpha_1)\frac{zH_{l,s}(\alpha_1+2)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2}$$
$$-2\gamma\alpha_1\frac{z[H_{l,s}(\alpha_1+1)f(z)]^2}{[H_{l,s}(\alpha_1)f(z)]^3},$$

is univalent in U, and

$$q(z)+\gamma zq'(z) \prec [1+\gamma(\alpha_1-1)]\frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2}+\gamma(1+\alpha_1)\frac{zH_{l,s}(\alpha_1+2)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2}$$
$$-2\gamma\alpha_1\frac{z[H_{l,s}(\alpha_1+1)f(z)]^2}{[H_{l,s}(\alpha_1)f(z)]^3},$$

then

$$q(z) \prec \frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2},$$

and q is the best subordinant.

Proof. The proof follows by applying Lemma 4. □

Taking $\alpha_1 = a > 0$, $\beta_1 = c > 0$, $\alpha_j = 1$ $(j = 2, \ldots, s+1)$ and $\beta_j = 1$ $(j = 2, \ldots, s)$ in Theorem 5, we have the following corollary which improve the result of Shanmugam et al. [16, Theorem 4.5].

Corollary 15. *Let q be convex univalent in U. Let $\gamma \in C$ with* $\operatorname{Re}\gamma > 0$. *If $f \in A$, $\frac{zL(a+1,c)f(z)}{[L(a,c)f(z)]^2} \in H[1,1] \cap Q$,*

$$[1+\gamma(a-1)]\frac{zL(a+1,c)f(z)}{[L(a,c)f(z)]^2}$$
$$+\gamma(1+a)\frac{zL(a+2,c)f(z)}{[L(a,c)f(z)]^2}-2\gamma a\frac{z[L(a+1,c)f(z)]^2}{[L(a,c)f(z)]^3}$$

is univalent in U, *and*

$$q(z)+\gamma zq'(z)\prec[1+\gamma(\alpha_1-1)]\frac{zL(a+1,c)f(z)}{[L(a,c)f(z)]^2}+\gamma(1+a)\frac{zL(a+2,c)f(z)}{[L(a,c)f(z)]^2}$$
$$-2\gamma a\frac{z[L(a+1,c)f(z)]^2}{[L(a,c)f(z)]^3},$$

then

$$q(z)\prec\frac{zL(a+1,c)f(z)}{[L(a,c)f(z)]^2},$$

and q *is the best subordinant.*

Combining Theorem 4 and Theorem 5, we get the following sandwich theorem.

Theorem 6. *Let* q_1 *be convex univalent in* U, $\gamma\in C$ *with* $Re\{\gamma\}>0$, q_2*be univalent in* U, $q_2(0)=1$ *and satisfies* (3.1). *If* $f\in A$, $\frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2}\in H[1,1]\cap Q$,

$$[1+\gamma(\alpha_1-1)]\frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2}$$
$$+\gamma(1+\alpha_1)\frac{zH_{l,s}(\alpha_1+2)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2}-2\gamma\alpha_1\frac{z[H_{l,s}(\alpha_1+1)f(z)]^2}{[H_{l,s}(\alpha_1)f(z)]^3}$$

is univalent in U, *and*

$$q_1(z)+\gamma zq_1'(z)\prec[1+\gamma(\alpha_1-1)]\frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2}+$$
$$\gamma(1+\alpha_1)\frac{zH_{l,s}(\alpha_1+2)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2}-2\gamma\alpha_1\frac{z[H_{l,s}(\alpha_1+1)f(z)]^2}{[H_{l,s}(\alpha_1)f(z)]^3}\prec q_2(z)+\gamma zq_2'(z),$$

then

$$q_1(z)\prec\frac{zH_{l,s}(\alpha_1+1)f(z)}{[H_{l,s}(\alpha_1)f(z)]^2}\prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

Combining Corollary 11 and Corollary 15, we get the following sandwich result which improves the result obtained by Shanmugam et al. [16, Corollary 4.6].

Corollary 16. *Let* $\gamma\in C$ *with* $\operatorname{Re}\gamma>0$, q_1 *be convex univalent in* U *and* q_2 *be univalent in* U, $q_2(0)=1$ *and satisfies* (3.1). *If* $f\in A$, $\frac{zL(a+1,c)f(z)}{[L(a,c)f(z)]^2}\in H[1,1]\cap Q$,

$$[1+\gamma(a-1)]\frac{zL(a+1,c)f(z)}{[L(a,c)f(z)]^2}$$
$$+\gamma(1+a)\frac{zL(a+2,c)f(z)}{[L(a,c)f(z)]^2}-2\gamma a\frac{z[L(a+1,c)f(z)]^2}{[L(a,c)f(z)]^3}$$

is univalent in U, *and*

$$q_1(z)+\gamma zq_1'(z) \prec [1+\gamma(\alpha_1-1)]\frac{zL(a+1,c)f(z)}{[L(a,c)f(z)]^2}+\gamma(1+a)\frac{zL(a+2,c)f(z)}{[L(a,c)f(z)]^2}$$

$$-2\gamma a\frac{z[L(a+1,c)f(z)]^2}{[L(a,c)f(z)]^3} \prec q_2(z)+\gamma zq_2'(z),$$

then

$$q_1(z) \prec \frac{zL(a+1,c)f(z)}{[L(a,c)f(z)]^2} \prec q_2(z),$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

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