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Circuminscribed polygons in a plane annulus

Abstract. Each oval and a natural number $n \geq 3$ generate an annulus which possesses the Poncelet's porism property. A necessary and sufficient condition of existence of circuminscribed n-gons in an annulus is given.

1. Introduction. Let C be an oval, i.e. a plane simple closed curve with a positive curvature, see [4]. We will consider C in the form

$$z(t) = p(t)\, e^{it} + \dot{p}(t)\, ie^{it} \quad \text{for } t \in [0, 2\pi], \tag{1.1}$$

where p is a fixed support function (the dot denotes the differentiation with respect to t). The function $R = p + \ddot{p}$ is the radius of curvature of C, see [6].

We associate with a fixed oval C a family $\mathcal{P}(C) = \{C_m : m > 0\}$ containing all parallel curves C_m to C; the support function p_m of C_m is given by the formula

$$p_m(t) = p(t) + m \quad \text{for } t \in [0, 2\pi], \tag{1.2}$$

see [6].

Moreover, we associate with C a second family $\mathcal{I}(C) = \{C_\alpha : 0 < \alpha < \pi\}$ where C_α is an α-isoptic of C. We recall that C_α is a locus of vertices of a fixed angle $\pi - \alpha$ formed by two support lines of the curve C, see [3].

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The parametric representation of C_α has the form

$$z_\alpha(t) = p(t) e^{it} + \frac{1}{\sin\alpha}\left(p(t+\alpha) - p(t)\cos\alpha\right) ie^{it} \quad \text{for } t \in [0, 2\pi]. \tag{1.3}$$

We will use the following functions:

$$\begin{cases} q(t) = z(t) - z(t+\alpha) \\ b(t) = p(t+\alpha)\sin\alpha + \dot{p}(t+\alpha)\cos\alpha - \dot{p}(t) \\ B(t) = p(t) - p(t+\alpha)\cos\alpha + \dot{p}(t+\alpha)\sin\alpha, \end{cases} \tag{1.4}$$

see [3].

We have

$$q = Be^{it} - bie^{it} \tag{1.5}$$

and

$$B > 0. \tag{1.6}$$

In this paper we will consider annuli formed by two ovals called convex annuli.

We will call that a convex annulus (C, D) formed by ovals C, D possess the Poncelet's porism property if through each point of the annulus pass circuminscribed n-gon (simultaneously inscribed in the outer oval and circumscribed on the inner oval). Basic information on the Poncelet's porism we can find in [1] but an extensive bibliography is given in [6], [2].

2. Convex isoptics of a parallel curve.

Theorem 2.1. *Let C be an oval and $\alpha \in (0, \pi)$. If the α-isoptic C_α is not convex then there exists a number $m^*(C, \alpha)$ such that if $m > m^*(C, \alpha)$ then each curve $C_{m,\alpha} \in \mathcal{I}(C_m)$ is an oval.*

Proof. Let us fix an oval C. If for a given $\alpha \in (0, \pi)$ the α-isoptic C_α of C is not convex, then we find a parallel curve C_m such that its α-isoptic $C_{m,\alpha}$ is convex.

Let us fix an arbitrary $m > 0$. The parametric equation of C_m has the form

$$Z(t) = (p(t) + m) e^{it} + \dot{p}(t) ie^{it} \quad \text{for } t \in [0, 2\pi]. \tag{2.1}$$

Let

$$Q(t) = Z(t) - Z(t+\alpha). \tag{2.2}$$

We have

$$Q = q + m(1 - \cos\alpha) e^{it} - m\sin\alpha ie^{it} \tag{2.3}$$

and

$$\dot{Q} = \dot{q} + m\sin\alpha e^{it} + m(1 - \cos\alpha) ie^{it}. \tag{2.4}$$

An α-isoptic $C_{m,a}$ of an oval C_m is convex if and only if

$$2\left|Q\right|^2 - \left[Q, \dot{Q}\right] \geq 0, \tag{2.5}$$

where $[\alpha+\beta i, \gamma+\delta i] = \alpha\delta - \beta\gamma$, see [3, Th. 5.2].

In view of (2.3) and (2.4) the left side of (2.5) has the form

$$\begin{aligned} 2\left|Q\right|^2 &- \left[Q, \dot{Q}\right] \\ &= 2\left(1-\cos\alpha\right)m^2 \\ &\quad + \left(\left(1-\cos\alpha\right)\left(3B - R - R_\alpha\right) + 3b\sin\alpha\right)m + 2\left|q\right|^2 - \left[q, \dot{q}\right] \\ &= 2\left(1-\cos\alpha\right)m^2 \\ &\quad + \left(\left(1-\cos\alpha\right)\left(3B - R - R_\alpha\right) + 3b\sin\alpha\right)m + k_{(\alpha)}\frac{\left|q\right|^3}{\sin\alpha}, \end{aligned} \tag{2.6}$$

where $k_{(\alpha)}$ is the curvature of the α-isoptic C_α and $R_\alpha(t) = R(t+\alpha)$, see [2, (5.7)].

Finally the α-isoptic of C_m is strictly convex if and only if the square inequality holds

$$\begin{aligned} &2\left(1-\cos\alpha\right)m^2 \\ &\quad + \left(\left(1-\cos\alpha\right)\left(3B - R - R_\alpha\right) + 3b\sin\alpha\right)m + k_{(\alpha)}\frac{\left|q\right|^3}{\sin\alpha} > 0. \end{aligned} \tag{2.7}$$

Let $a_1(t) \leq a_2(t)$ denotes roots of square equation associated with (2.7) if

$$\Delta_\alpha(t) = \left(\left(1-\cos\alpha\right)\left(3B-R-R_\alpha\right) + 3b\sin\alpha\right)^2 - 8\left(1-\cos\alpha\right)k_{(\alpha)}\frac{\left|q\right|^3}{\sin\alpha} \geq 0.$$

Thus we can define a number

$$m^*(C,\alpha) = \max\left\{a_2(t) : t \in [0, 2\pi], \quad \Delta_\alpha(t) \geq 0\right\}. \tag{2.8}$$

If $m > m^*(C,\alpha)$ then $C_{m,\alpha}$ is strictly convex. We recall that an α-isoptic of an oval is a curve of the class C^2 [3, Th. 5.1]. □

3. On the Poncelet's porism property. Mozgawa in [5] proved that for a given oval C there exist ovals C_{in} and C_{out}, inside and outside of C, such that the pairs (C, C_{in}) and (C_{out}, C) has the Poncelet's porism property for almost any natural number n.

In this paper we can solve the modified problem, namely: For a given oval C and a fixed natural number $n \geq 3$ find a convex annulus generated by C and n possessing the Poncelet's porism property.

It follows from the previous section that the following theorem holds:

Theorem 3.1. *Let $n \geq 3$ be a fixed natural number and $\alpha = \frac{2\pi}{n}$. Let T be an arbitrary affine transformation. For each oval C and $m > m^*(C,\alpha)$ each convex annulus $\left(T(C_m), T(C_{m,\alpha})\right)$ where $C_m \in \mathcal{P}(C)$ and $C_{m,\alpha} \in \mathcal{I}(C)$ possess the Poncelet's porism property.*

4. Existence of circuminscribed n-gons in a convex annulus. Let (C, D) be a convex annulus. If the inner oval C is given by (1.1) then the outer oval D will be considered in the form

$$w(t) = z(t) + \lambda(t)\, ie^{it} \quad \text{for } t \in [0, 2\pi]\,. \tag{4.1}$$

where λ is a positive-valued function.

We note that a fixed point $w(t)$ of D belongs to some α-isoptic C_α. From (1.3) and (4.1) we get the implicit equation for α, namely

$$(\dot{p}(t) + \lambda(t)) \sin\alpha - p(t+\alpha) + p(t)\cos\alpha = 0. \tag{4.2}$$

Let $F(t, \alpha) = (\dot{p}(t) + \lambda(t)) \sin\alpha - p(t+\alpha) + p(t)\cos\alpha$. Then we have

$$\begin{aligned}\frac{\partial F}{\partial \alpha} &= (\dot{p} + \lambda)\cos\alpha - \dot{p}_\alpha - p\sin\alpha \\ &= \frac{-B + [p(t)\cos\alpha - p(t+\alpha) + (\dot{p}(t) + \lambda(t))\sin\alpha]\cos\alpha}{\sin\alpha} \\ &= \frac{-B}{\sin\alpha} < 0.\end{aligned}$$

Applying the implicit function theorem we have a differentiable function $\alpha(t)$. Let

$$\varphi(t) = t + \alpha(t) \tag{4.3}$$

and $\varphi^{[1]} = \varphi$, $\varphi^{[n]} = \varphi \circ \varphi^{[n-1]}$ for $n = 2, 3, \ldots$. The following theorems hold:

Theorem 4.1. *There exists circuminscribed n-gon in a convex annulus (C, D) if and only if the equation $\varphi^{[n]}(t) - t - 2\pi = 0$ has a solution in the interval* $[0, 2\pi]$.

Theorem 4.2. *A convex annulus (C, D) possess the Poncelet's porism property if and only if for some natural number $n \geq 3$ the function $\varphi^{[n]}(t) - t - 2\pi$ vanish.*

Applications of Theorem 4.2 will be given in the other paper.

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