DOI: 10.2478/v10062-008-0015-1

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXII, 2008 SECTIO A 143–147

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Properties of harmonic conjugates

ABSTRACT. We give a new proof of Hardy and Littlewood theorem concerning harmonic conjugates of functions u such that $\int_{\mathbb{D}} |u(z)|^p dA(z) < \infty$, $0 < p \le 1$. We also obtain an inequality for integral means of such harmonic functions u.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and dA be the Lebesgue measure normalized so that $A(\mathbb{D}) = 1$. The harmonic Hardy space h^p, $0 < p < \infty$, consists of all real-valued functions u harmonic in $\mathbb{D}$ whose integral means

$$M_p(r,u) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}$$

are bounded. The harmonic Bergman space a^p is the collection of all real-valued harmonic functions u in $\mathbb{D}$ for which the integral

$$||u||_p^p = \int_{\mathbb{D}} |u(z)|^p dA(z)$$

is finite. For a real-valued function u harmonic in $\mathbb{D}$ we define the harmonic conjugate as the function v with $v(0) = 0$ such that $f = u + iv$ is analytic in $\mathbb{D}$. By the theorem of M. Riesz, if $1 < p < \infty$ and $u \in h^p$, then $v \in h^p$ and $M_p(r,v) \le CM_p(r,u)$ where C depends only on p. For $0 < p \le 1$ or $p = \infty$ the theorem fails.

2000 *Mathematics Subject Classification.* 30H05, 32A36.
Key words and phrases. Hardy and Littlewood theorem, harmonic conjugate, a^p space.

It follows immediately from the theorem of M. Riesz that for every p in the range $1 < p < \infty$ if $u \in a^p$, then $v \in a^p$ and $||v||_p \leq C||u||_p$. However, in the space a^p the last inequality holds also for $0 < p \leq 1$. This result was first stated by Hardy and Littlewood [4] and its proof was indicated there. Thus the following theorem holds.

Theorem HL. *Let $0 < p < \infty$. If $u \in a^p$, then its conjugate $v \in a^p$ and $||v||_p \leq C||u||_p$, where C depends only on p.*

In [4] Watanabe presented the proof of the above theorem, when $0 < p \leq 1$. There are some gaps and the proof seems to be incomplete. For example the inequality in line 9 from the above on page 53 is not proved. We note that in the case when $0 < p < 1$ and u is harmonic in $\mathbb{D}$ the integral mean $M_p(r, u)$ need not be monotonically increasing function of r. Moreover, the application of Lemma 4 in [1] at the end of the proof is not explained. In this paper we give a complete detailed proof of Theorem HL for the case $0 < p \leq 1$, shorter than that in [4]. Throughout this paper C denotes a general positive constant which may differ from line to line.

Proof of Theorem HL for the case when $0 < p \leq 1$. Let $f = u + iv$ be analytic in $\mathbb{D}$ and assume that $v(0) = 0$. We start with the following inequality proved in [1] p. 411.

$$\sigma|zf'(z)| \leq \eta^{-1}\left(|u(r+h,\theta)| + |u(r,\theta+h)| + 2|u(r,\theta)|\right) + Ar\mu\sigma\eta, \tag{1}$$

where $z = re^{i\theta}$, $0 < r < 1$, $u(r,\theta) = u(re^{i\theta}), \sigma = \sigma(r) = \sqrt{r} - r$, $h = \eta\sigma$, $A = \sum_{m=2}^{\infty} 2^m \eta^{m-2} = 4/(1-2\eta)$, η is any positive number less than $\frac{1}{4}$. Moreover, $\mu = \mu(r,\theta) = \max\limits_{\gamma} |f'(z)|$ and γ denotes the circle centered at the point $re^{i\theta}$ and the radius σ.

Since $0 < p \leq 1$, we get from (1)

$$\begin{aligned} &\sigma(r)^p \frac{1}{2\pi}\int_0^{2\pi} r^p |f'(re^{i\theta})|^p d\theta \\ &\qquad \leq \eta^{-p}\left(\frac{1}{2\pi}\int_0^{2\pi} |u(r+h,\theta)|^p d\theta\right. \\ &\qquad\qquad \left. + \frac{1}{2\pi}\int_0^{2\pi} |u(r,\theta+h)|^p d\theta + 2^p \frac{1}{2\pi}\int_0^{2\pi} |u(r,\theta)|^p d\theta\right) \\ &\qquad + (A\sigma\eta)^p \frac{1}{2\pi}\int_0^{2\pi} (r\mu)^p d\theta. \end{aligned} \tag{2}$$

It was shown in [1] p. 411 that

$$\frac{1}{2\pi}\int_0^{2\pi} (r\mu)^p d\theta \leq C\frac{1}{2\pi}\int_0^{2\pi} r^{\frac{p}{4}} |f'(r^{\frac{1}{4}} e^{i\theta})|^p d\theta.$$

Moreover, an easy calculation shows that $\sigma(r) \le 4\sigma\big(r^{\frac{1}{4}}\big)$. Now multiplying both sides of inequality (2) by $2r$ and integrating with respect r give

$$\begin{aligned}
&\frac{1}{\pi}\int_0^1\int_0^{2\pi}\sigma(r)^p r^p|f'(re^{i\theta})|^p d\theta r dr \\
&\quad\le \eta^{-p}\left(\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(r+h,\theta)|^p d\theta r dr \right.\\
&\qquad\qquad\left. + (2^p+1)\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(r,\theta)|^p d\theta r dr\right) \\
&\qquad + C\eta^p\frac{1}{\pi}\int_0^1\int_0^{2\pi}\sigma(r^{\frac{1}{4}})^p r^{\frac{p}{4}}|f'(r^{\frac{1}{4}}e^{i\theta})|^p d\theta r dr.
\end{aligned}$$

Substituting $t^4 = r$ in the last integral yields

$$\begin{aligned}
&\frac{1}{\pi}\int_0^1\int_0^{2\pi}\sigma(r)^p r^p|f'(re^{i\theta})|^p d\theta r dr \\
&\quad\le \eta^{-p}\left(\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(r+h,\theta)|^p d\theta r dr \right.\\
&\qquad\qquad\left. + (2^p+1)\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(r,\theta)|^p d\theta r dr\right) \\
&\quad + C\eta^p\frac{1}{\pi}\int_0^1\int_0^{2\pi}\sigma(t)^p t^p|f'(te^{i\theta})|^p d\theta t dt.
\end{aligned} \tag{3}$$

It is clear that $r+h = r+\eta(\sqrt{r}-r) < 1$ on $0<r<1$ and $0<\eta<\frac{1}{4}$. Moreover, the function $g(r) = r+\eta(\sqrt{r}-r)$ is increasing in the interval $0<r<1$. Substituting $r+h=t^2$ in the first integral on the right hand side of (3) we get

$$\begin{aligned}
&\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(r+h,\theta)|^p d\theta r dr \\
&\quad= \frac{2}{1(1-\eta)}\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(t^2,\theta)|^p\left(\frac{-\eta+\sqrt{\eta^2+4(1-\eta)t^2}}{2(1-\eta)}\right)^2 \\
&\qquad\qquad\times\left(\frac{-\eta}{\sqrt{\eta^2+4(1-\eta)t^2}}+1\right) t d\theta dt \\
&\quad\le \frac{4}{2(1-\eta)}\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(t^2,\theta)|^p\left(\frac{-\eta+\sqrt{\eta^2+4(1-\eta)t^2}}{2(1-\eta)}\right)^2 \\
&\qquad\qquad\times\left(\frac{-\eta}{2-\eta}+1\right) d\theta t dt
\end{aligned}$$

$$= \frac{4}{2-\eta}\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(t^2,\theta)|^p\left(\frac{-\eta+\sqrt{\eta^2+4(1-\eta)t^2}}{2(1-\eta)}\right)^2 d\theta t dt$$
$$\leq \frac{4}{2-\eta}\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(t^2,\theta)|^p\left(\frac{-\eta+\eta+\sqrt{4(1-\eta)t^2}}{2(1-\eta)}\right)^2 d\theta t dt$$
$$= \frac{4}{(2-\eta)(1-\eta)}\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(t^2,\theta)|^p t^2 d\theta t dt$$
$$= \frac{2}{(2-\eta)(1-\eta)}\frac{1}{\pi}\int_0^1\int_0^{2\pi}|u(t,\theta)|^p d\theta t dt.$$

By the assumption $u \in a^p$ and (3) we get

$$\frac{1}{\pi}\int_0^1\int_0^{2\pi}\sigma(r)^p r^p|f'(re^{i\theta})|^p d\theta r dr$$
$$\leq \frac{1}{\eta^p}\left(\frac{2}{(2-\eta)(1-\eta)}+2^p+1\right)||u||_{a^p}^p$$
$$+ C\eta^p\frac{1}{\pi}\int_0^1\int_0^{2\pi}\sigma(t)^p t^p|f'(te^{i\theta})|^p d\theta t dt.$$

Now choosing η so that $\eta < C^{-\frac{1}{p}}$ we get

$$(4)\qquad (1-C\eta^p)\frac{1}{\pi}\int_0^1\int_0^{2\pi}\sigma(r)^p r^p|f'(re^{i\theta})|^p d\theta r dr \leq C||u||_{a^p}^p.$$

We note that the convergence of the above integral implies the convergence of

$$\frac{1}{\pi}\int_0^1\int_0^{2\pi}(1-r)^p|f'(re^{i\theta})|^p d\theta r dr,$$

which means that $f \in A^p$, see e.g. Lemma 4 in [4]. □

Corollary. *If* $u \in a^p,\ u(0)=0,\ 0<p\leq 1,$ *then*

$$M_p(r,u) \leq C\frac{||u||_{a^p}}{(1-r)^{\frac{1}{p}}},$$

where a constant C *depends only on* p.

Proof. Let f and σ be as in our proof of Theorem HL and assume that $f(0)=0$. It is clear that the function σ is monotonically increasing in $\left(0,\frac{1}{4}\right)$ and monotonically decreasing in $\left(\frac{1}{4},1\right)$. Since $M_p(r,f')$ is increasing function of r on $(0,1)$, using the Chebyshev inequality (see e.g. [3]) we get

$$\begin{aligned}
&\int_0^1\int_0^{2\pi}\sigma(r)^p r^p|f'(re^{i\theta})|^p r d\theta dr\\
&\quad=\int_0^{\frac14}\int_0^{2\pi}\sigma(r)^p r^p|f'(re^{i\theta})|^p r d\theta dr+\int_{\frac14}^1\int_0^{2\pi}\sigma(r)^p r^p|f'(re^{i\theta})|^p r d\theta dr\\
&\quad\geq C\int_0^{\frac14}\int_0^{2\pi}|f'(re^{i\theta})|^p r d\theta dr+\frac{1}{8^p}\int_{\frac14}^1\int_0^{2\pi}\big(1-\sqrt{r}\big)^p|f'(re^{i\theta})|^p r d\theta dr\\
&\quad\geq C\int_0^1\int_0^{2\pi}(1-r)^p|f'(re^{i\theta})|^p dr\theta dr\geq C\int_0^1\int_0^{2\pi}|f(re^{i\theta})|^p r d\theta dr,
\end{aligned}$$

where the last inequality follows from e.g. Lemma 4 in [4]. Thus

$$M_p^p(r,u)(1-r)\leq M_p^p(r,f)(1-r)\leq\int_r^1\frac{1}{2\pi}\int_0^{2\pi}|f(te^{i\theta})|^p d\theta t dt\leq C||u||_{a^p}^p\ .$$

$\square$

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Received December 4, 2007