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Differential sandwich theorems for analytic functions defined by Cho–Kwon–Srivastava operator

Abstract. By making use of Cho–Kwon–Srivastava operator, we obtain some subordinations and superordinations results for certain normalized analytic functions.

1. Introduction. Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z : |z| < 1\}$ and $H(a, n)$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in C).$$

For simplicity, let $H[a] = H[a, 1]$. Also, let A be the subclass of the functions $f \in H(U)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For $f, g \in H(U)$, we say that the function f is subordinate to g, or the function g is superordinate to f, if there exists a Schwarz function w, i.e., $w \in H(U)$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. This subordination is usually denoted by $f(z) \prec g(z)$. It

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is well known that, if the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (cf., e.g., [7], see also [4]).

Supposing that p, h are two analytic functions in U, let

$$\varphi(r, s, t; z) : C^3 \times U \to C.$$

If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order subordination

$$h(z) \prec \varphi\big(p(z), zp'(z), z^2p''(z); z\big), \tag{1.2}$$

then p is called to be a solution of the differential superordination (1.2). A function $q \in H(U)$ is called a subordinant of (1.2), if $q(z) \prec p(z)$ for all the functions $p(z)$ satisfying (1.2). A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (1.2), is called the best subordinant (cf., e.g., [7], see also [4]).

Recently, Miller and Mocanu [8] obtained sufficient conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi\big(p(z), zp'(z),\, z^2p''(z); z\big) \Rightarrow q(z) \prec p(z).$$

For functions $f_j(z) \in A$, given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2),$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z) \quad (z \in U).$$

In terms of the Pochhammer symbol $(\theta)_n$ given by

$$(\theta)_n = \begin{cases} 1, & (n = 0) \\ \theta(\theta + 1) \ldots (\theta + n - 1), & (n \in N = \{1, 2, \ldots\}), \end{cases}$$

we now define a function $\varphi(a, c; z)$ by

$$\varphi(a, c; z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \tag{1.3}$$

$(a \in R;\ c \in R \backslash Z_0^-;\ Z_0^- = \{0, -1, -2, \ldots\};\ z \in U)$.

With the aid of the function $\varphi(a, c; z)$ defined by (1.3), we consider a function $\varphi^*(a, c; z)$ given by the following convolution

$$\varphi(a, c; z) * \varphi^*(a, c; z) = \frac{z}{(1 - z)^{\lambda+1}} \quad (\lambda > -1;\ z \in U)$$

which yields the following family of linear operators $I^\lambda(a, c)$:

$$I^\lambda(a, c) f(z) = \phi^*(a, c; z) * f(z) \quad (a, c \in R \backslash Z_0^-;\ \lambda > -1;\ z \in U). \tag{1.4}$$

For a function $f(z) \in A$, given by (1.1), it is easily seen from (1.4) that

$$(1.5) \qquad I^{\lambda}(a,c)f(z) = z + \sum_{n=2}^{\infty} \frac{(c)_{n-1}(\lambda+1)_{n-1}}{(a)_{n-1}(1)_{n-1}} a_n z^n \quad (z \in U),$$

which readily yields the following

$$(1.6) \qquad z(I^{\lambda}(a,c)f(z))' = (\lambda+1)I^{\lambda+1}(a,c)f(z) - \lambda I^{\lambda}(a,c)f(z)$$

and

$$(1.7) \qquad z(I^{\lambda}(a+1,c)f(z))' = aI^{\lambda}(a,c)f(z) - (a-1)I^{\lambda}(a+1,c)f(z).$$

The operator $I^{\lambda}(a,c)$ was introduced and studied by Cho et al. [5].

We also observe that:

(i) $I^0(1,1)f(z) = I^1(2,1)f(z) = f(z)$, $I^1(1,1)f(z) = zf'(z)$, $I^2(1,1)f(z) = \frac{1}{2}(2zf'(z) + z^2 f''(z))$;

(ii) $I^{\mu}(\mu+2,1)f(z) = F_{\mu}(f)(z)$ $(\mu > -1)$, where

$$F_{\mu}(f)(z) = \frac{\mu+1}{z^{\mu}} \int_0^z t^{\mu-1} f(t)dt \quad \text{(see [2]);}$$

(iii) $I^0(n+1,1)f(z) = I_n f(z)$ $(n \in N_0 = N \cup \{0\})$ (Noor integral operator, see [11]);

(iv) $I^{\lambda}(\mu+2,1)f(z) = I_{\lambda,\mu}f(z)$ $(\lambda > -1;\ \mu > -2)$ (Choi–Saigo–Srivastava operator see [6]).

Recently many authors ([1], [9], [10] and [12]) have used the results of Bulboacă [3] and shown some sufficient conditions applying first order differential subordinations and superordinations.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions $f(z)$, $g(z)$ in U such that $I^{\lambda}(a,c)g(z) \neq 0$ for $0 < |z| < 1$ and satisfy

$$q_1(z) \prec \frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)g(z)} \prec q_2(z),$$

where q_1, q_2 are given univalent functions in U. Also, we obtain the number of known results as their special cases.

2. Definitions and preliminaries.

In order to prove our results, we shall make use of the following known results.

Definition 1 ([8]). Denote by Q, the set of all functions f that are analytic and injective on $\overline{U}\backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

Lemma 1 ([7]). *Let q be univalent in the unit disk U and let θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set*

$$\psi(z) = zq^{'}(z)\varphi(q(z)) \text{ and } h(z) = \theta(q(z)) + \psi(z).$$

Suppose that

(i) *$\psi(z)$ is starlike univalent in U,*

(ii) $\operatorname{Re}\left\{\dfrac{zh^{'}(z)}{\psi(z)}\right\} > 0, \quad z \in U.$

If p is analytic in U with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta\big(p(z)\big) + zp^{'}(z)\varphi(p(z)\big) \prec \theta\big(q(z)\big) + zq^{'}(z)\varphi(q(z)\big), \tag{2.1}$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Lemma 2 ([3]). *Let q be convex univalent in the unit disk U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\operatorname{Re}\left\{\dfrac{\theta^{'}(q(z))}{\varphi(q(z))}\right\} > 0, \quad z \in U,$

(ii) *$\psi(z) = zq^{'}(z)\varphi(q(z))$ is starlike univalent in U.*

If $p \in H[q(0), 1] \cap Q$ with $p(U) \subseteq D$ and $\theta(p(z)) + zp^{'}(z)\varphi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq^{'}(z)\varphi(q(z)) \prec \theta(p(z)) + zp^{'}(z)\varphi(p(z)), \tag{2.2}$$

then

$$q(z) \prec p(z),$$

and q is the best subordinant of (2.2).

3. Subordination results. Using Lemma 1, we first prove the following theorem.

Theorem 1. *Let $\alpha \neq 0$, $\beta > 0$ and $q(z)$ be convex univalent in U with $q(0) = 1$. Further assume that*

$$\operatorname{Re}\left\{\frac{\beta - \alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq^{''}(z)}{q^{'}(z)}\right)\right\} > 0 \quad (z \in U). \tag{3.1}$$

If $f, g \in A$ satisfy

$$\gamma(f, g, \alpha, \beta) \prec (\beta - \alpha)q(z) + \alpha q^2(z) + \alpha zq^{'}(z), \tag{3.2}$$

where

$$\gamma(f,g,\alpha,\beta) = (\beta-2\alpha)\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)g(z)} + \alpha\left(\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)g(z)}\right)^2$$
$$(3.3)\qquad + \alpha(\lambda+2)\frac{I^{\lambda+2}(a,c)f(z)}{I^{\lambda}(a,c)g(z)}$$
$$- \alpha(\lambda+1)\frac{I^{\lambda+1}(a,c)g(z)}{I^{\lambda}(a,c)g(z)}\left(\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)g(z)}\right),$$

then

$$\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)g(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$(3.4)\qquad p(z) = \frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)g(z)} \quad (z \in U).$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$. Therefore, differentiating (3.4) logarithmically with respect to z and using the identity (1.6) in the resulting equation, we have

$$\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)g(z)}\left[\beta - 2\alpha + \alpha\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)g(z)}\right.$$
$$(3.5)\qquad \left. + \alpha(\lambda+2)\frac{I^{\lambda+2}(a,c)f(z)}{I^{\lambda+1}(a,c)f(z)} - \alpha(\lambda+1)\frac{I^{\lambda+1}(a,c)g(z)}{I^{\lambda}(a,c)g(z)}\right]$$
$$= (\beta-\alpha)p(z) + \alpha p^2(z) + \alpha z p'(z).$$

By using (3.5) in (3.2), we have

$$(3.6)\quad (\beta-\alpha)p(z) + \alpha p^2(z) + \alpha z p'(z) \prec (\beta-\alpha)q(z) + \alpha q^2(z) + \alpha z q'(z).$$

By setting

$$\theta(w) = \alpha w^2 + (\beta-\alpha)w \quad \text{and} \quad \varphi(w) = \alpha,$$

we can easily observe that $\theta(w)$ and $\varphi(w)$ are analytic in $C\backslash\{0\}$ and that $\varphi(w) \neq 0$. Hence the result now follows by using Lemma 1. □

Remark 1. Putting $\lambda = 0$, $a = c = 1$ and taking $f(z) \equiv g(z)$ $(z \in U)$ in Theorem 1, we obtain the result obtained by Murugusundaramoorthy and Magesh [10, Corollary 2.9].

Putting $f(z) \equiv g(z)$ $(z \in U)$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let $\alpha \neq 0$, $\beta > 0$ and q be convex univalent in U with $q(0) = 1$ and (3.1) holds true. If $f \in A$ satisfies*

$$(\beta - 2\alpha)\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)f(z)} + \alpha(\lambda+2)\frac{I^{\lambda+2}(a,c)f(z)}{I^{\lambda}(a,c)f(z)} - \alpha\lambda\left(\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)f(z)}\right)^2$$
$$\prec (\beta-\alpha)q(z) + \alpha q^2(z) + \alpha z q^{'}(z),$$

then

$$\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)f(z)} \prec q(z),$$

and q is the best dominant.

Putting $a = \mu + 2$ $(\mu > -2)$ and $c = 1$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Let $\alpha \neq 0$, $\beta > 0$ and q be convex univalent in U with $q(0) = 1$ and (3.1) holds true. If $f, g \in A$ satisfy*

$$\gamma_1(f,g,\alpha,\beta) \prec (\beta-\alpha)q(z) + \alpha q^2(z) + \alpha z q^{'}(z),$$

where

$$\begin{aligned}\gamma_1(f,g,\alpha,\beta) &= (\beta-2\alpha)\frac{I_{\lambda+1,\mu}f(z)}{I_{\lambda,\mu}g(z)} + \alpha\left(\frac{I_{\lambda+1,\mu}f(z)}{I_{\lambda,\mu}g(z)}\right)^2 \\ &+ \alpha(\lambda+2)\frac{I_{\lambda+2,\mu}f(z)}{I_{\lambda,\mu}g(z)} - \alpha(\lambda+1)\frac{I_{\lambda+1,\mu}g(z)}{I_{\lambda,\mu}g(z)}\left(\frac{I_{\lambda+1,\mu}f(z)}{I_{\lambda,\mu}g(z)}\right),\end{aligned} \tag{3.7}$$

then

$$\frac{I_{\lambda+1,\mu}f(z)}{I_{\lambda,\mu}g(z)} \prec q(z),$$

and q is the best dominant.

Putting $a = \mu + 2$ $(\mu > -1)$, $c = 1$ and $\lambda = \mu$ in Theorem 1, we obtain the following corollary.

Corollary 3. *Let $\alpha \neq 0$, $\beta > 0$ and q be convex univalent in U with $q(0) = 1$ and (3.1) holds true. If $f, g \in A$ satisfy*

$$\gamma_2(f,g,\alpha,\beta) \prec (\beta-\alpha)q(z) + \alpha q^2(z) + \alpha z q^{'}(z),$$

where

$$\begin{aligned}\gamma_2(f,g,\alpha,\beta) &= (\beta-\alpha+\alpha\mu)\frac{f(z)}{F_{\mu}(g)(z)} + \alpha\left(\frac{f(z)}{F_{\mu}(g)(z)}\right)^2 \\ &+ \alpha\frac{zf^{'}(z)}{F_{\mu}(g)(z)} - \alpha(\mu+1)\frac{g(z)}{F_{\mu}(g)(z)}\frac{f(z)}{F_{\mu}(g)(z)},\end{aligned} \tag{3.8}$$

then

$$\frac{f(z)}{F_{\mu}(g)(z)} \prec q(z),$$

and q is the best dominant.

Putting $f(z) \equiv g(z)$ $(z \in U)$ in Corollary 3, we obtain the following corollary.

Corollary 4. *Let* $\alpha \neq 0$, $\beta > 0$ *and* q *be convex univalent in* U *with* $q(0) = 1$ *and* (3.1) *holds true. If* $f \in A$ *satisfies*

$$\gamma_3(f, \alpha, \beta) \prec (\beta - \alpha)q(z) + \alpha q^2(z) + \alpha z q'(z),$$

where

$$\gamma_3(f, \alpha, \beta) = (\beta - \alpha + \alpha\mu)\frac{f(z)}{F_\mu(f)(z)} + \alpha\frac{zf'(z)}{F_\mu(f)(z)} - \alpha\mu\left(\frac{f(z)}{F_\mu(f)(z)}\right)^2, \tag{3.9}$$

then

$$\frac{f(z)}{F_\mu(f)(z)} \prec q(z) \quad (\mu > -1),$$

and q *is the best dominant.*

4. Superordination and sandwich results.

Theorem 2. *Let* $\alpha \neq 0$ *and* $\beta > 0$*. Let* q *be convex univalent in* U *with* $q(0) = 1$*. Assume that*

$$\operatorname{Re}\{q(z)\} \geq \operatorname{Re}\left\{\frac{\alpha - \beta}{2\alpha}\right\}. \tag{4.1}$$

Let $f, g \in A$, $\dfrac{I^{\lambda+1}(a,c)f(z)}{I^\lambda(a,c)g(z)} \in H[q(0), 1] \cap Q$*. Let* $\gamma(f, g, \alpha, \beta)$ *be univalent in* U *and*

$$(\beta - \alpha)q(z) + \alpha q^2(z) + \alpha z q'(z) \prec \gamma(f, g, \alpha, \beta), \tag{4.2}$$

where $\gamma(f, g, \alpha, \beta)$ *is given by* (3.3)*, then*

$$q(z) \prec \frac{I^{\lambda+1}(a,c)f(z)}{I^\lambda(a,c)g(z)} \tag{4.3}$$

and q *is the best subordinant.*

Proof. Let $p(z)$ be defined by (3.4). Therefore, differentiating (3.4) with respect to z and using the identity (1.6) in the resulting equation, we have

$$\gamma(f, g, \alpha, \beta) = (\beta - \alpha)p(z) + \alpha p^2(z) + \alpha z p'(z),$$

then

$$(\beta - \alpha)q(z) + aq^2(z) + \alpha z q'(z) \prec (\beta - \alpha)p(z) + \alpha p^2(z) + \alpha z p'(z).$$

By setting $\theta(w) = \alpha w^2 + (\beta - \alpha)w$ and $\varphi(w) = \alpha$, it is easily observed that $\theta(w)$ is analytic in C. Also, $\varphi(w)$ is analytic in $C\backslash\{0\}$ and that $\varphi(w) \neq 0$. Since $q(z)$ is convex univalent, it follows that

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\} = \operatorname{Re}\left\{\frac{\beta - \alpha}{\alpha} + 2q(z)\right\} > 0 \quad (z \in U).$$

Now Theorem 2 follows by applying Lemma 2. □

Putting $f(z) \equiv g(z)$ $(z \in U)$ in Theorem 2, we obtain the following corollary.

Corollary 5. *Let* $\alpha \neq 0$, $\beta \geq 1$ *and* q *be convex univalent in* U *with* $q(0) = 1$ *and* (4.1) *holds true. Let* $f \in A$, $\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)f(z)} \in H[q(0),1] \cap Q$. *Let*

$$\gamma(f,\alpha,\beta) = (\beta - 2\alpha)\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)f(z)} + \alpha(\lambda+2)\frac{I^{\lambda+2}(a,c)f(z)}{I^{\lambda}(a,c)f(z)} - \alpha\lambda\left(\frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)f(z)}\right)^2,$$

be univalent in U *and*

$$(\beta - \alpha)q(z) + aq^2(z) + \alpha zq^{'}(z) \prec \gamma(f,\alpha,\beta),$$

then

$$q(z) \prec \frac{I^{\lambda+1}(a,c)f(z)}{I^{\lambda}(a,c)f(z)},$$

and q *is the best subordinant.*

Putting $a = \mu + 2$ $(\mu > -2)$ and $c = 1$ in Theorem 2, we obtain the following corollary.

Corollary 6. *Let* $\alpha \neq 0$, $\beta \geq 1$ *and* q *be convex univalent in* U *with* $q(0) = 1$ *and* (4.1) *holds true. Let* $f, g \in A$, $\frac{I_{\lambda+1,\mu}f(z)}{I_{\lambda,\mu}g(z)} \in H[q(0),1] \cap Q$. *Let* $\gamma_1(f,g,\alpha,\beta)$ *be univalent in* U *and*

$$(\beta - \alpha)q(z) + \alpha q^2(z) + \alpha zq^{'}(z) \prec \gamma_1(f,g,\alpha,\beta),$$

where $\gamma_1(f,g,\alpha,\beta)$ *is given by* (3.7), *then*

$$q(z) \prec \frac{I_{\lambda+1,\mu}f(z)}{I_{\lambda,\mu}g(z)},$$

and q *is the best subordinant.*

Putting $a = \mu + 2$ $(\mu > -1)$, $c = 1$ and $\lambda = \mu$ in Theorem 2, we obtain the following corollary.

Corollary 7. *Let* $\alpha \neq 0$, $\beta \geq 1$ *and* q *be convex univalent in* U *with* $q(0) = 1$ *and* (4.1) *holds true. Let* $f, g \in A$, $\frac{f(z)}{F_{\mu}(g)(z)} \in H[q(0),1] \cap Q$. *Let* $\gamma_2(f,g,\alpha,\beta)$ *be univalent in* U *and*

$$(\beta - \alpha)q(z) + \alpha q^2(z) + \alpha zq^{'}(z) \prec \gamma_2(f,g,\alpha,\beta),$$

where $\gamma_2(f,g,\alpha,\beta)$ *is given by* (3.8), *then*

$$q(z) \prec \frac{f(z)}{F_\mu(g)(z)},$$

and q *is the best subordinant.*

Putting $f(z) \equiv g(z)$ $(z \in U)$ in Corollary 7, we obtain the following corollary.

Corollary 8. *Let* $\alpha \neq 0$, $\beta \geq 1$ *and* q *be convex univalent in* U *with* $q(0)=1$ *and* (4.1) *holds true. Let* $f \in A$, $\frac{f(z)}{F_\mu(f)(z)} \in H[q(0),1] \cap Q$. *Let* $\gamma_3(f,\alpha,\beta)$ *be univalent in* U *and*

$$(\beta-\alpha)q(z)+\alpha q^2(z)+\alpha z q^{'}(z) \prec \gamma_3(f,\alpha,\beta),$$

where $\gamma_3(f,\alpha,\beta)$ *is given by* (3.9), *then*

$$q(z) \prec \frac{f(z)}{F_\mu(f)(z)} \quad (\mu > -1),$$

and q *is the best subordinant.*

We conclude this section by stating the following sandwich result.

Theorem 3. *Let* q_1 *and* q_2 *be convex univalent in* U, $\alpha \neq 0$ *and* $\beta \geq 1$. *Suppose* q_2 *satisfies* (3.1) *and* q_1 *satisfies* (4.1). *Moreover, suppose*

$$\frac{I^{\lambda+1}(a,c)f(z)}{I^\lambda(a,c)g(z)} \in H[1,1] \cap Q$$

and $\gamma(f,g,\alpha,\beta)$ *is univalent in* U. *If* $f,g \in A$ *satisfy*

$$(\beta-\alpha)q_1(z)+\alpha q_1^2(z)+azq_1^{'}(z) \prec \gamma(f,g,\alpha,\beta) \prec (\beta-\alpha)q_2(z)+\alpha q_2^2(z)+azq_2^{'}(z),$$

where $\gamma(f,g,\alpha,\beta)$ *is given by* (3.3), *then*

$$q_1(z) \prec \frac{I^{\lambda+1}(a,c)f(z)}{I^\lambda(a,c)g(z)} \prec q_2(z)$$

and q_1, q_2 *are, respectively, the best subordinant and the best dominant.*

By making use of Corollaries 2 and 6, we obtain the following corollary.

Corollary 9. *Let* q_1 *and* q_2 *be convex univalent in* U, $\alpha \neq 0$ *and* $\beta \geq 1$. *Suppose* q_2 *satisfies* (3.1) *and* q_1 *satisfies* (4.1). *Moreover, suppose*

$$\frac{I_{\lambda+1,\mu}(a,c)f(z)}{I_{\lambda,\mu}(a,c)g(z)} \in H[1,1] \cap Q$$

and $\gamma_1(f, g, \alpha, \beta)$ *is univalent in* U*. If* $f, g \in A$ *satisfy*

$$(\beta-\alpha)q_1(z)+\alpha q_1^2(z)+\alpha z q_1^{'}(z) \prec \gamma_1(f,g,\alpha,\beta) \prec (\beta-\alpha)q_2(z)+\alpha q_2^2(z)+\alpha z q_2^{'}(z),$$

where $\gamma_1(f, g, \alpha, \beta)$ *is given by* (3.7)*, then*

$$q_1(z) \prec \frac{I_{\lambda+1,\mu}f(z)}{I_{\lambda,\mu}g(z)} \prec q_2(z) \quad (\mu > -2)$$

and q_1,q_2 *are, respectively, the best subordinant and the best dominant.*

By making use of Corollaries 3 and 7, we obtain the following corollary.

Corollary 10. *Let* q_1 *and* q_2 *be convex univalent in* U*,* $\alpha \neq 0$ *and* $\beta \geq 1$*. Suppose* q_2 *satisfies* (3.1) *and* q_1 *satisfies* (4.1)*. Moreover, suppose*

$$\frac{f(z)}{F_\mu(g)(z)} \in H[1,1] \cap Q$$

and $\gamma_2(f, g, \alpha, \beta)$ *is univalent in* U*. If* $f, g \in A$ *satisfy*

$$(\beta-\alpha)q_1(z)+\alpha q_1^2(z)+\alpha z q_1^{'}(z) \prec \gamma_2(f,g,\alpha,\beta) \prec (\beta-\alpha)q_2(z)+\alpha q_2^2(z)+\alpha z q_2^{'}(z),$$

where $\gamma_2(f, g, \alpha, \beta)$ *is given by* (3.8)*, then*

$$q_1(z) \prec \frac{f(z)}{F_\mu(g)(z)} \prec q_2(z) \quad (\mu > -1)$$

and q_1*,* q_2 *are, respectively, the best subordinant and the best dominant.*

By making use of Corollaries 4 and 8, we obtain the following corollary.

Corollary 11. *Let* q_1 *and* q_2 *be convex univalent in* U*,* $\alpha \neq 0$ *and* $\beta \geq 1$*. Suppose* q_2 *satisfies* (3.1) *and* q_1 *satisfies* (4.1)*. Moreover, suppose*

$$\frac{f(z)}{F_\mu(f)(z)} \in H[1,1] \cap Q$$

and $\gamma_3(f, \alpha, \beta)$ *is univalent in* U*. If* $f \in A$ *satisfies*

$$(\beta-\alpha)q_1(z)+\alpha q_1^2(z)+\alpha z q_1^{'}(z) \prec \gamma_3(f,\alpha,\beta) \prec (\beta-\alpha)q_2(z)+\alpha q_2^2(z)+\alpha z q_2^{'}(z),$$

where $\gamma_3(f, \alpha, \beta)$ *is given by* (3.9)*, then*

$$q_1(z) \prec \frac{f(z)}{F_\mu(f)(z)} \prec q_2(z) \quad (\mu > -1)$$

and q_1*,* q_2 *are, respectively, the best subordinant and the best dominant.*

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