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Almost symplectic structures on the linear frame bundle from linear connection

Abstract. We describe all $\mathcal{M}f_m$-natural operators $S\colon Q \rightsquigarrow Symp\, P^1$ transforming classical linear connections ∇ on m-dimensional manifolds M into almost symplectic structures $S(\nabla)$ on the linear frame bundle P^1M over M.

Let V be a real vector space of even dimension. A bilinear form $\varpi\colon V \times V \to \mathbb{R}$ is called a symplectic form if it is antisymmetric and nondegenerate, i.e. it satisfies

$$\varpi(v,v) = 0 \text{ for all } v \in V \text{ and if } \varpi(v,u) = 0 \text{ for all } v \in V, \text{ then } u = 0.$$

A vector space V is a symplectic vector space if it is equipped with a symplectic form, [1].

Let $\mathcal{M}f_m$ denote the category of m-dimensional manifolds and their embeddings and $\mathcal{FM}$ denote the category of fibred manifolds and fibred maps between them.

For any m-dimensional manifold M we have the linear frame bundle $P^1M = invJ^1_0(\mathbb{R}^m, M)$ of the manifold M. This is a principal bundle with corresponding Lie group $GL(m) = G^1_m = invJ^1_0(\mathbb{R}^m, \mathbb{R}^m)_0$, which acts on P^1M on the right via compositions of jets. Every map $\psi\colon M_1 \to M_2$ from the category $\mathcal{M}f_m$ induces a map $P^1\psi\colon P^1M_1 \to P^1M_2$ by $P^1\psi(j^1_0\varphi) = j^1_0(\psi \circ \varphi)$, where $\varphi\colon \mathbb{R}^m \to M_1$ is a map from the category $\mathcal{M}f_m$. The correspondence $P^1\colon \mathcal{M}f_m \to \mathcal{FM}$ is a bundle functor in the sense of [3].

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For any $2n$-dimensional manifold N we have an almost symplectic structures bundle $Symp\,(N) = \bigcup_{y\in N} \widetilde{Symp}\,(T_yN)$ over the manifold N, where $\widetilde{Symp}\,(T_yN)$ denotes the set of symplectic forms $\varpi\colon T_yN \times T_yN \to \mathbb{R}$ on the tangent space T_yN. The bundle $Symp\,(N)$ is a subbundle (but not vector subbundle) of a vector bundle $T^*N \otimes T^*N$ of tensors of type $(0,2)$ over N. Sections $\Omega\colon N \to Symp\,(N)$ are called almost symplectic structures on the manifold N. Every embedding $\psi\colon N_1 \to N_2$ induces a fibred map $Symp\,(\psi)\colon Symp\,(N_1) \to Symp\,(N_2)$ being restriction of $T^*\psi \otimes T^*\psi\colon T^*N_1 \otimes T^*N_1 \to T^*N_2 \otimes T^*N_2$ to $Symp\,(N)$. The correspondence $Symp\colon \mathcal{M}f_{2n} \to \mathcal{FM}$ is a bundle functor in the sense of [3].

Let M be an m-dimensional manifold. We have the classical linear connection bundle $QM := (id_{T^*M} \otimes \pi^1)^{-1}(id_{TM}) \subset T^*M \otimes J^1TM$ of the manifold M, where $\pi^1\colon J^1TM \to TM$ is the projection of the first jet prolongation $J^1TM = \{j^1_xX\colon X \in \mathfrak{X}(M), x \in M\}$ of the tangent bundle TM of the manifold M. Sections $\tilde{\nabla}\colon M \to QM$ correspond bijectively to classical linear connections on M. Every embedding $f\colon M_1 \to M_2$ induces a fibred map $Qf\colon QM_1 \to QM_2$ covering f. The correspondence $Q\colon \mathcal{M}f_m \to \mathcal{FM}$ is a bundle functor in the sense of [3].

Let $\{A^{j*}_i\}$, $i,j = 1,\dots,m$ be the standard basis in $\mathfrak{gl}(m) = \mathcal{L}ie(GL(m))$.

For a principal fibre bundle P^1M the action of group $GL(m)$ on P^1M induces a homomorphism σ of Lie algebra $\mathfrak{gl}(m)$ of group $GL(m)$ into Lie algebra $\mathfrak{X}(P^1M)$ of vector fields on P^1M. For every $A \in \mathfrak{gl}(m)$, a vector field $A^* = \sigma(A)$ is called the fundamental vector field corresponding to A. Since the action of group $GL(m)$ on P^1M sends each fibre into itself, therefore A^*_u is tangent to the fibre at each $u \in P^1M$, [2].

Let ∇ be a classical linear connection on m-dimensional manifold M. For every $\xi \in \mathbb{R}^m$ we define the standard horizontal vector field $B(\xi)$ on P^1M as follows. For each $u \in P^1M$, $u\colon \mathbb{R}^m \to T_{\pi(u)}M$, a vector $(B(\xi))_u$ is the unique horizontal vector at u such that $T\pi((B(\xi))_u) = u(\xi)$, where $\pi\colon P^1M \to M$, [2].

The canonical form θ of bundle P^1M is $\mathbb{R}^m$-valued 1-form on P^1M defined by

$$\theta(X) = u^{-1}(T\pi(X)) \qquad \text{for } X \in T_u(P^1M),$$

where $\pi\colon P^1M \to M$ and $u\colon \mathbb{R}^m \to T_{\pi(u)}(M)$, [2].

For a given connection ∇ on P^1M we define a 1-form ω on P^1M with values in Lie algebra $\mathfrak{gl}(m)$ of group $GL(m)$ as follows. For each $X \in T_u(P^1M)$ we define $\omega(X)$ to be the unique $A \in \mathfrak{gl}(m)$ such that $(A^*)_u$ is equal to the vertical component of vector X. The form ω is called the connection form of the given connection ∇, [2].

Let $B_1,\dots,B_m$ be the standard horizontal vector fields corresponding to basic vectors $e_1,\dots,e_m$ of space $\mathbb{R}^m$ and let $\{A^{j*}_i\}$ be fundamental vector fields corresponding to basic vectors $\{A^j_i\}$ of Lie algebra $\mathfrak{gl}(m)$. It is easy to verify that $\{B_l, A_i^{j*}\}$ and $\{\theta^i, \omega_j^i\}$ are dual to each other, i.e. they satisfy

$$\theta^k(B_l) = \delta_l^k, \quad \theta^k(A_i^{j*}) = 0,$$
$$\omega_r^k(B_l) = 0, \quad \omega_r^k(A_i^{j*}) = \delta_i^k \delta_r^j,$$

where θ^i are components of the canonical 1-form and ω_j^i are components of the connection form.

Proposition 1 ([2]). *The $m^2 + m$ vector fields $\{B_k, A_i^{j^*}; i, j, k = 1, \ldots, m\}$ define an absolute parallelism in the bundle P^1M.*

The following definition of a natural operator is particular case of an idea of natural operator which was considered in [3].

Definition 1. An $\mathcal{M}f_m$-natural operator $S\colon Q \rightsquigarrow Symp\, P^1$ is a family of $\mathcal{M}f_m$-invariant regular operators $S = (S_M)$

$$S_M\colon \underline{Q}(M) \to \underline{Symp}\,(P^1M)$$

for any manifold M from the category $\mathcal{M}f_m$, where $\underline{Q}(M)$ is the set of all linear connections on the manifold M (sections of $Q(M) \to M$) and $\underline{Symp}\,(P^1M)$ is the set of all almost symplectic structures on P^1M (sections of $Symp\,(P^1M) \to P^1M$). The invariance means that if $\nabla_1 \in \underline{Q}(M_1)$ and $\nabla_2 \in \underline{Q}(M_2)$ are ψ-related by $\psi\colon M_1 \to M_2$, that is $Q(\psi) \circ \nabla_1 = \nabla_2 \circ \psi$, then $S(\nabla_1)$ and $S(\nabla_2)$ are $P^1\psi$-related, that is $Symp\,(P^1\psi) \circ S(\nabla_1) = S(\nabla_2) \circ P^1\psi$. The regularity means that smoothly parametrized families of classical linear connections are transformed by S on smoothly parametrized families of almost symplectic structures.

In the present note we will classify all natural operators S and obtained result will be modification of result in [4].

Remark 1. In [4] there were described geometric constructions on higher order frame bundles P^rM. In the present paper we describe only case of linear frame bundle P^1M. The generalization of this problem for P^rM is not possible, because dimension of P^rM for $r > 1$ does not have to be even.

For given connection $\nabla \in \underline{Q}(M)$ with respect to the global basis of vector fields $\{B_k, A_i^{j*}\}$ on P^1M we have a canonical (in ∇) fibred diffeomorphism

$$K_\nabla\colon P^1M \times \widetilde{Symp}\,(\mathbb{R}^{m^2+m}) \to Symp\,(P^1M)$$

covering id_{P^1M} defined by the condition that the matrix of map $K_\nabla(u(x), \varpi)$ in the basis $\{B_k(\nabla)(u(x)), A_i^{j*}(u(x))\}$ is the same as the one of the symplectic form ϖ in the canonical basis of space $\mathbb{R}^{m^2+m}$.

Let $Z^s = J^s_0(Q(\mathbb{R}^m))$, $s = 0, 1, \ldots, \infty$ be the set of s-jets $j^s_0\nabla$ of all classical linear connections ∇ on $\mathbb{R}^m$ satisfying

$$\sum_{j,k=1}^{m} \nabla^i_{jk}(x)x^j x^k = 0 \quad \text{for } i = 1, \ldots, m,$$

it means that the usual coordinates $x^1, \ldots, x^m$ on $\mathbb{R}^m$ are ∇-normal with center $0 \in \mathbb{R}^m$.

Example 1. General construction: Let $\mu\colon Z^\infty \to \widetilde{Symp}\,(\mathbb{R}^{m^2+m})$ be a map satisfying the following local finite determination property.

For any $\rho \in Z^\infty$ we can find an open neighborhood $U \subset Z^\infty$ of jet ρ, a natural number s and a smooth map $f\colon \pi_s(U) \to \widetilde{Symp}\,(\mathbb{R}^{m^2+m})$ such that $\mu = f \circ \pi_s$ on U, where $\pi_s\colon Z^\infty \to Z^s$ is the jet projection. (A simple example of such μ is $\mu = f \circ \pi_s$ for smooth $f\colon Z^s \to \widetilde{Symp}\,(\mathbb{R}^{m^2+m})$ and for finite number s.)

Given a classical linear connection ∇ on an m-dimensional manifold M we define an almost symplectic structure $S^{\langle\mu\rangle}(\nabla)$ on P^1M as follows. Let $u(x) \in (P^1M)_x$, $x \in M$. Choose a ∇-normal coordinate system ψ on M with center x such that $P^1\psi(u(x)) = l^0 = j^1_0(id_{\mathbb{R}^m})$. Such a coordinate system ψ exists. Then $germ_x(\psi)$ is uniquely determined. We put

$$S^{\langle\mu\rangle}(\nabla)_{u(x)} = Symp\,(P^1(\psi^{-1}))(K_{\psi_*\nabla}(l^0, \mu(j^\infty_0(\psi_*\nabla)))).$$

Since $germ_x(\psi)$ is uniquely determined, then above definition is correct. The family $S^{\langle\mu\rangle}\colon Q \rightsquigarrow Symp\,P^1$ is an $\mathcal{M}f_m$-natural operator.

Theorem 1. *Any $\mathcal{M}f_m$-natural operator $S\colon Q \rightsquigarrow Symp\,P^1$ is of the form $S^{<\mu>}$ for some uniquely determined (by S) function $\mu\colon Z^\infty \to \widetilde{Symp}\,(\mathbb{R}^{m+m^2})$ satisfying local finite determination property.*

Proof. Let $S\colon Q \rightsquigarrow Symp\,P^1$ be an $\mathcal{M}f_m$-natural operator. Define $\mu\colon Z^\infty \to \widetilde{Symp}\,(\mathbb{R}^{m+m^2})$ by

$$(l^0, \mu(j^\infty_0\nabla)) = K^{-1}_\nabla(S(\nabla)(l^0)).$$

Then by non-linear Peetre theorem, [3], μ satisfies local finite determination property. Then by definitions of μ and $S^{<\mu>}$ we have that $S(\nabla)(l^0) = S^{<\mu>}(\nabla)(l^0)$ for any classical linear connection ∇ on $\mathbb{R}^m$ such that the identity map $id_{\mathbb{R}^m}$ is a ∇-normal coordinate system with center $0 \in \mathbb{R}^m$. Then by the invariance of S and $S^{<\mu>}$ with respect to normal coordinates we deduce that $S = S^{<\mu>}$. □

Remark 2. Symplectic geometry methods are key ingredients in the study of dynamical systems, mathematical physics, analytical mechanics, differential geometry, [1], [5].

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