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On maximum modulus for the derivative of a polynomial

Abstract. If $P(z)$ is a polynomial of degree n, having all its zeros in the disk $|z| \le k$, $k \ge 1$, then it was shown by Govil [Proc. Amer. Math. Soc. **41**, no. 2 (1973), 543–546] that

$$\max_{|z|=1} \left|P'(z)\right| \ge \frac{n}{1+k^n} \max_{|z|=1} |P(z)|\,.$$

In this paper, we obtain generalization as well as improvement of above inequality for the polynomial of the type $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu \le n$. Also we generalize a result due to Dewan and Mir [Southeast Asian Bull. Math. **31** (2007), 691–695] in this direction.

1. Introduction and statement of results. If $P(z)$ is a polynomial of degree n and $P'(z)$ its derivative, then according to a famous result known as Bernstein's inequality (for reference see [1]), we have

$$\max_{|z|=1} \left|P'(z)\right| \le n \max_{|z|=1} |P(z)|\,. \tag{1.1}$$

For the polynomial $P(z)$, it is well known as a simple consequence of maximum modulus principle (for reference see [7, p. 158, problem 269]) that for $R \ge 1$,

$$\max_{|z|=R} |P(z)| \le R^n \max_{|z|=1} |P(z)|\,. \tag{1.2}$$

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Both the inequalities (1.1) and (1.2) are sharp and equality holds for $P(z) = \alpha z^n$, where $|\alpha| = 1$.

Turán [9] considered that if $P(z)$ is a polynomial of degree n, having all its zeros in $|z| \le 1$, then

$$\max_{|z|=1} \left|P'(z)\right| \ge \frac{n}{2} \max_{|z|=1} |P(z)| . \tag{1.3}$$

The result is best possible and equality holds for $P(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

As a generalization of inequality (1.3), Govil [3] proved the following result.

Theorem A. *If $P(z)$ is a polynomial of degree n, having all its zeros in the disk $|z| \le k$, $k \ge 1$, then*

$$\max_{|z|=1} \left|P'(z)\right| \ge \frac{n}{1+k^n} \max_{|z|=1} |P(z)| . \tag{1.4}$$

The result is best possible and equality holds for $P(z) = (z^n + k^n)$.

For the polynomial not vanishing in $|z| < k$, $k \le 1$, Govil [4] proved that if $P(z)$ has all its zeros on $|z| = k$, $k \le 1$, then

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |P(z)| . \tag{1.5}$$

While seeking for the better bound of the inequality (1.5), recently Dewan and Mir [2] proved the following result under the same hypothesis.

Theorem B. *If $P(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n, having all its zeros on $|z| = k$, $k \le 1$, then*

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{k^n} \left\{ \frac{n\,|c_n|\,k^2 + |c_{n-1}|}{n\,|c_n|\,(1+k^2) + 2\,|c_{n-1}|} \right\} \max_{|z|=1} |P(z)| . \tag{1.6}$$

In this paper, we consider a class of polynomials $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu \le n$ and generalize as well as improve upon Theorem A and also generalize Theorem B by proving the following results.

Theorem 1. *If $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ is a polynomial of degree n, having all its zeros in the disk $|z| \le k$, $k \ge 1$, then*

$$\max_{|z|=1} \left|P'(z)\right| \ge \frac{n}{1+k^{n-\mu+1}} \max_{|z|=1} |P(z)| . \tag{1.7}$$

The result is best possible and equality holds for

$$P(z) = \left(z^{n-\mu+1} + k^{n-\mu+1}\right)^{\frac{n}{n-\mu+1}} .$$

Remark 1. If we take $\mu = 1$ in Theorem 1, then inequality (1.7) reduces to inequality (1.4) due to Govil [3].

Theorem 2. *If* $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \le \mu < n$ *is a polynomial of degree* n, *having all its zeros on* $|z| = k$, $k \le 1$, *then*

$$\max_{|z|=1} \left|P'(z)\right| \le \frac{n}{k^{n-\mu+1}} \left(\frac{n\,|c_n|\,k^{2\mu} + \mu\,|c_{n-\mu}|\,k^{\mu-1}}{\mu\,|c_{n-\mu}|\,(1+k^{\mu-1}) + n\,|c_n|\,k^{\mu-1}(1+k^{\mu+1})} \right) \max_{|z|=1} |P(z)|\,.$$

Remark 2. If we take $\mu = 1$ in Theorem 2, then the above inequality reduces to the inequality (1.6) due to Dewan and Mir [2].

Theorem 3. *If* $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ *is a polynomial of degree* n, *having all its zeros in the disk* $|z| \le k$, $k \ge 1$, *then*

$$\max_{|z|=1} \left|P'(z)\right| \ge \frac{n}{1+k^{n-\mu+1}} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \right\}. \tag{1.8}$$

The result is best possible and equality holds for

$$P(z) = \left(z^{n-\mu+1} + k^{n-\mu+1}\right)^{\frac{n}{n-\mu+1}}.$$

If we choose $\mu = 1$ in Theorem 3, then inequality (1.8) reduces to following result due to Govil [5].

Corollary 1. *If* $P(z)$ *is a polynomial of degree* n, *having all its zeros in the disk* $|z| \le k$, $k \ge 1$, *then*

$$\max_{|z|=1} \left|P'(z)\right| \ge \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \right\}.$$

The result is best possible and equality holds for $P(z) = z^n + k^n$.

2. Lemmas. We need the following lemmas for the proofs of these theorems.

Lemma 1. *If* $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ *is a polynomial of degree* n, *having all its zeros in the disk* $|z| \le k$, $k \ge 1$, *then for* $|z| = 1$

$$k^{n+\mu-3} \left|Q'(z)\right| \le \left|P'(k^2 z)\right|, \tag{2.1}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Proof of Lemma 1. Since the polynomial $P(z)$ has all its zeros in $|z| \le k$, $k \ge 1$, therefore the polynomial $F(z) = P(kz)$ has all its zeros in the unit disk $|z| \le 1$. Now if $G(z) \equiv z^n \overline{F(1/\overline{z})} \equiv z^n \overline{P(k/\overline{z})} \equiv k^n Q(z/k)$, then all the zeros of $G(z)$ lie in $|z| \ge 1$. Since $|F(z)| = |G(z)|$ on $|z| = 1$, it follows by maximum modulus principle that $|G(z)| \le |F(z)|$ on $|z| \ge 1$. Hence for every complex number λ with $|\lambda| > 1$, it follows by Rouche's theorem that the polynomial $G(z) - \lambda F(z)$ has all its zeros in $|z| < 1$. By Gauss–Lucas theorem the polynomial $G'(z) - \lambda F'(z)$ has all its zeros in $|z| < 1$, which implies

$$\left|G'(z)\right| \le \left|F'(z)\right| \quad \text{for } |z| \ge 1. \tag{2.2}$$

Substituting for $F(z)$ and $G(z)$ in (2.2), we get

$$k^{n-1}\left|Q'(z/k)\right| \le k\left|P'(kz)\right| \quad \text{for } |z| \ge 1. \tag{2.3}$$

Since $c_1 = c_2 = \cdots = c_{\mu-1} = 0$, from (2.3), we get

$$k^{n-1}\left|Q'(z/k)\right| \le k^{\mu}\left|\sum_{\nu=\mu}^{n} \nu c_\nu (kz)^{\nu-\mu}\right| \quad \text{for } |z| \ge 1. \tag{2.4}$$

In fact (2.4) holds for $|z| = 1$. But $\sum_{\nu=\mu}^{n} \nu c_\nu (kz)^{\nu-\mu} \neq 0$ in $|z| > 1$, by maximum modulus principle it also holds for $|z| > 1$. Taking kz instead of z in (2.4), we have

$$k^{n-1}\left|Q'(z)\right| \le k^{\mu}\left|\sum_{\nu=\mu}^{n} \nu c_\nu (k^2 z)^{\nu-\mu}\right| \quad \text{for } |z| \ge 1/k.$$

In particular,

$$k^{n-1}\left|Q'(z)\right| \le k^{\mu}\left|\sum_{\nu=\mu}^{n} \nu c_\nu (k^2 z)^{\nu-\mu}\right| \quad \text{for } |z| = 1,$$

this implies

$$k^{n-1}\left|Q'(z)\right| \le k^{2-\mu}\left|\sum_{\nu=\mu}^{n} \nu c_\nu (k^2 z)^{\nu-1}\right| \quad \text{for } |z| = 1.$$

Consequently

$$k^{n+\mu-3}\left|Q'(z)\right| \le \left|P'(k^2 z)\right| \quad \text{for } |z| = 1.$$

This completes the proof of Lemma 1. □

Lemma 2. *If $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ is a polynomial of degree n, having all its zeros in the disk $|z| \le k$, $k \ge 1$, then*

$$\max_{|z|=1}\left|Q'(z)\right| \le k^{n-\mu+1}\max_{|z|=1}\left|P'(z)\right|,$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Proof of Lemma 2. By Lemma 1, we have

$$\max_{|z|=1}\left|Q'(z)\right| \le \frac{1}{k^{n+\mu-3}}\max_{|z|=k^2}\left|P'(z)\right|. \tag{2.5}$$

Using inequality (1.2) for the polynomial $P'(z)$ with $R = k^2 \ge 1$, we have

$$\max_{|z|=k^2}\left|P'(z)\right| \le k^{2n-2}\max_{|z|=1}\left|P'(z)\right|.$$

Combining this with (2.5), the lemma follows. □

Lemma 3. *If* $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ *is a polynomial of degree* n*, having no zero in the disk* $|z| < k$, $k \leq 1$*, then*

$$k^{n-\mu+1} \max_{|z|=1} \left|P'(z)\right| \leq \max_{|z|=1} \left|Q'(z)\right|,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Proof of Lemma 3. If $P(z)$ has no zero in $|z| < k$, $k \leq 1$, then $Q(z) = z^n \overline{P(1/\overline{z})}$ has all its zeros in $|z| \leq 1/k$, $1/k \geq 1$. Thus applying Lemma 2 to the polynomial $Q(z)$, we get

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{1}{k^{n-\mu+1}} \max_{|z|=1} \left|Q'(z)\right|,$$

and the lemma follows. □

Lemma 4. *If* $P(z)$ *is a polynomial of degree* n*, then for* $|z| = 1$

$$\left|P'(z)\right| + \left|Q'(z)\right| \leq n \max_{|z|=1} |P(z)|,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

The above lemma is a special case of a result due to Govil and Rahman [6].

Lemma 5. *If* $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \leq \mu \leq n$ *is a polynomial of degree* n*, having no zero in the disk* $|z| < k$, $k \geq 1$*, then for* $|z| = 1$

$$k^{\mu+1} \left\{ \frac{\mu |c_\mu| k^{\mu-1} + n |c_0|}{n |c_0| + \mu |c_\mu| k^{\mu+1}} \right\} \left|P'(z)\right| \leq \left|Q'(z)\right|$$

and

$$\frac{\mu}{n} \left| \frac{c_\mu}{c_0} \right| k^\mu \leq 1,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

The above lemma was given by Qazi [8, Remark and proof of Lemma 1].

Lemma 6. *If* $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$ *is a polynomial of degree* n*, having all its zeros on* $|z| = k$, $k \leq 1$*, then for* $|z| = 1$

$$k^{\mu-1} \left\{ \frac{n |c_n| k^{\mu+1} + \mu |c_{n-\mu}|}{\mu |c_{n-\mu}| + n |c_n| k^{\mu-1}} \right\} \left|P'(z)\right| \geq \left|Q'(z)\right| \tag{2.6}$$

and

$$\frac{\mu}{n} \left| \frac{c_{n-\mu}}{c_n} \right| \leq k^\mu, \tag{2.7}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Proof of Lemma 6. Since $P(z)$ has all its zeros on $|z| = k$, $k \leq 1$, therefore $Q(z) = z^n \overline{P(1/\overline{z})}$ has all its zeros on $|z| = 1/k$, $1/k \geq 1$. Now applying Lemma 5 to polynomial $Q(z)$ and result follows. □

The following lemma is due to Govil [3].

Lemma 7. *If $P(z)$ is a polynomial of degree n and $P(z) \equiv Q(z)$, then for $|z| = 1$*

$$\max_{|z|=1} \left|P'(z)\right| = \frac{n}{2} \max_{|z|=1} |P(z)|,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

3. Proofs of the theorems.

Proof of Theorem 1. For every ϵ with $|\epsilon| = 1$, the polynomial $P^*(z) = \frac{1}{2}(P(z) + \epsilon Q(z))$ satisfies $P^*(z) \equiv z^n \overline{P^*(1/\overline{z})}$, hence by Lemma 7, we have

$$\max_{|z|=1} \left|P'(z) + \epsilon Q'(z)\right| = \frac{n}{2} \max_{|z|=1} |P(z) + \epsilon Q(z)|.$$

This implies

$$\max_{|z|=1} \left|P'(z)\right| + \max_{|z|=1} \left|Q'(z)\right| \geq \frac{n}{2} \max_{|z|=1} |P(z) + \epsilon Q(z)|.$$

Choosing the argument of ϵ on right hand side, we get

$$\max_{|z|=1} \left|P'(z)\right| + \max_{|z|=1} \left|Q'(z)\right| \geq n \max_{|z|=1} |P(z)|.$$

Which further on applying Lemma 2, gives

$$\max_{|z|=1} \left|P'(z)\right| + k^{n-\mu+1} \max_{|z|=1} \left|P'(z)\right| \geq n \max_{|z|=1} |P(z)|$$

and the theorem follows. □

Proof of Theorem 2. Let z_0 be a point on $|z| = 1$, such that $|Q'(z_0)| = \max_{|z|=1} |Q'(z)|$, then by Lemma 4, it follows that

$$\left|P'(z_0)\right| + \max_{|z|=1} \left|Q'(z)\right| \leq n \max_{|z|=1} |P(z)|. \tag{3.1}$$

Combining inequality (3.1) with Lemma 6, we get

$$\frac{1}{k^{\mu-1}} \left(\frac{\mu\,|c_{n-\mu}| + n\,|c_n|\,k^{\mu-1}}{n\,|c_n|\,k^{\mu+1} + \mu\,|c_{n-\mu}|}\right) \left|Q'(z_0)\right| + \max_{|z|=1} \left|Q'(z)\right| \leq n \max_{|z|=1} |P(z)|,$$

which implies

$$\begin{aligned} &\left(\frac{\mu\,|c_{n-\mu}|\,(1 + k^{\mu-1}) + n\,|c_n|\,k^{\mu-1}(1 + k^{\mu+1})}{n\,|c_n|\,k^{2\mu} + \mu\,|c_{n-\mu}|\,k^{\mu-1}}\right) \max_{|z|=1} \left|Q'(z)\right| \\ &\leq n \max_{|z|=1} |P(z)|. \end{aligned} \tag{3.2}$$

Inequality (3.2), when combined with Lemma 3, gives

$$\begin{aligned} &k^{n-\mu+1} \left(\frac{\mu\,|c_{n-\mu}|\,(1 + k^{\mu-1}) + n\,|c_n|\,k^{\mu-1}(1 + k^{\mu+1})}{n\,|c_n|\,k^{2\mu} + \mu\,|c_{n-\mu}|\,k^{\mu-1}}\right) \max_{|z|=1} \left|P'(z)\right| \\ &\leq n \max_{|z|=1} |P(z)|. \end{aligned}$$

This implies

$$\max_{|z|=1}\left|P'(z)\right|$$
$$\leq \frac{n}{k^{n-\mu+1}}\left(\frac{n\left|c_n\right|k^{2\mu}+\mu\left|c_{n-\mu}\right|k^{\mu-1}}{\mu\left|c_{n-\mu}\right|(1+k^{\mu-1})+n\left|c_n\right|k^{\mu-1}(1+k^{\mu+1})}\right)\max_{|z|=1}|P(z)|,$$

which completes the proof of Theorem 2. □

Proof of Theorem 3. If $m=\min_{|z|=k}|P(z)|$, then for every α with $|\alpha|<1$, the polynomial $P(z)+\alpha m$ has all its zeros in $|z|\leq k$, $k\geq 1$. This is clear if $P(z)$ has a zero on $|z|=k$, because in that case $m=0$ and therefore $P(z)+\alpha m=P(z)$. In case $P(z)$ has no zero on $|z|=k$, then for every α with $|\alpha|<1$, we have $|P(z)|>m|\alpha|$ on $|z|=k$ and on applying Rouche's theorem the result will follow. Thus $P(z)+\alpha m$ has all its zeros in $|z|\leq k$, $k\geq 1$ and hence, applying Theorem 1 to $P(z)+\alpha m$, we get

$$\max_{|z|=1}\left|P'(z)\right|\geq \frac{n}{1+k^{n-\mu+1}}\max_{|z|=1}|P(z)+\alpha m|.$$

Now choosing argument of α on the right hand side and letting $|\alpha|\to 1$, we get

$$\max_{|z|=1}\left|P'(z)\right|\geq \frac{n}{1+k^{n-\mu+1}}\left\{\max_{|z|=1}|P(z)|+\min_{|z|=k}|P(z)|\right\}.$$

This completes the proof of Theorem 3. □

Remark 3. For $\mu=n$ Theorems 1, 2 and 3 hold if polynomial satisfies the condition $|c_0|\leq k\,|c_n|$.

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