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Inclusion and neighborhood properties of certain subclasses of p -valent functions of complex order defined by convolution

ABSTRACT. In this paper we introduce and investigate three new subclasses of p -valent analytic functions by using the linear operator $D_{\lambda,p}^m(f * g)(z)$. The various results obtained here for each of these function classes include coefficient bounds, distortion inequalities and associated inclusion relations for (n, θ) -neighborhoods of subclasses of analytic and multivalent functions with negative coefficients, which are defined by means of a non-homogenous differential equation.

1. Introduction. Let $A_p(n)$ denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{k=n}^{\infty} a_k z^k \quad (n > p; p, n \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic and p -valent in the open unit disk $U = \{z : |z| < 1\}$. The Hadamard product (or convolution) of the functions $f(z)$ given by (1.1), and $g(z) \in A_p(n)$ given by

$$(1.2) \quad g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \quad (n > p; p, n \in \mathbb{N})$$

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is defined by

$$(1.3) \quad (f * g)(z) = z^p + \sum_{k=n}^{\infty} a_k b_k z^k = (g * f)(z).$$

For functions $f, g \in A_p(n)$, we define the linear operator $D_{\lambda,p}^m : A_p(n) \rightarrow A_p(n)$ ($\lambda \geq 0$; $p, n \in \mathbb{N}$; $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) by

$$(1.4) \quad D_{\lambda,p}^0(f * g)(z) = (f * g)(z),$$

$$(1.5) \quad D_{\lambda,p}^1(f * g)(z) = D_{\lambda,p}(f * g)(z) = (1 - \lambda)(f * g)(z) + \frac{\lambda z}{p}(f * g)'(z)$$

and (in general)

$$(1.6) \quad \begin{aligned} D_{\lambda,p}^m(f * g)(z) &= D_{\lambda,p}(D_{\lambda,p}^{m-1}(f * g)(z)) \\ &= (1 - \lambda)D_{\lambda,p}^{m-1}(f * g)(z) + \frac{\lambda z}{p} \left(D_{\lambda,p}^{m-1}(f * g) \right)'(z) \\ &= z^p + \sum_{k=n}^{\infty} \left[\frac{p + \lambda(k-p)}{p} \right]^m a_k b_k z^k \end{aligned}$$

($\lambda \geq 0$; $p, n \in \mathbb{N}$; $m \in \mathbb{N}_0$; $z \in U$).

The operator $D_{\lambda,1}^m(f * g)(z) = D_{\lambda}^m(f * g)(z)$ was introduced by Aouf and Seoudy [6].

We note that

(i) for $\lambda = 1$ and $b_k = 1$ (or $g(z) = \frac{z^p}{1-z}$), $D_{\lambda,p}^m(f * g)(z) = D_p^m f(z)$, where the operator D_p^m is the p -valent Salagean operator introduced and studied by Aouf and Mostafa [5], Kamali and Orhan [11] and Orhan and Kiziltunc [13];

(ii) for $b_k = 1$ (or $g(z) = \frac{z^p}{1-z}$), $D_{\lambda,p}^m(f * g)(z) = D_{\lambda,p}^m f(z)$, where the operator $D_{\lambda,p}^m$ was introduced and studied by El-Ashwah and Aouf [8].

For a function $f(z) \in A_p(n)$, we have

$$(1.7) \quad (D_{\lambda,p}^m(f * g)(z))^{(q)} = \delta(p, q) z^{p-q} + \sum_{k=n}^{\infty} \delta(k, q) \left[\frac{p + \lambda(k-p)}{p} \right]^m a_k b_k z^{k-q},$$

($\lambda \geq 0$; $p, n \in \mathbb{N}$; $q, m \in \mathbb{N}_0$; $p > q$; $z \in U$), where

$$(1.8) \quad \delta(p, q) = \begin{cases} 1, & (q = 0), \\ p(p-1) \dots (p-q+1), & (q \neq 0). \end{cases}$$

We denote by $T_p(n)$ the subclass of $A_p(n)$ consisting of functions of the form

$$(1.9) \quad f(z) = z^p - \sum_{k=n}^{\infty} a_k z^k \quad (n > p; a_k \geq 0; p, n \in \mathbb{N}).$$

For a given function $g(z) \in A_p(n)$ defined by

$$(1.10) \quad g(z) = z^p + \sum_{k=n}^{\infty} b_k z^k \quad (b_k > 0; n > p; p, n \in \mathbb{N}),$$

we now introduce a new subclass $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ of the class $T_p(n)$ of p -valently analytic functions, which consists of functions $f(z) \in T_p(n)$ satisfying the inequality

$$(1.11) \quad \left| \frac{1}{b} \left\{ \frac{z(D_{\lambda,p}^m(f*g)(z))^{(q+1)} + \gamma z^2(D_{\lambda,p}^m(f*g)(z))^{(q+2)}}{(1-\gamma)(D_{\lambda,p}^m(f*g)(z))^{(q)} + \gamma z(D_{\lambda,p}^m(f*g)(z))^{(q+1)}} - (p-q) \right\} \right| < \beta$$

($\lambda \geq 0; p, n \in \mathbb{N}; q, m \in \mathbb{N}_0; 0 \leq \gamma \leq 1; p > q; 0 < \beta \leq 1; b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}; z \in U$).

We note that

(1) $C_0^q(g(z); n, 0, p, \lambda, 1, b) = S_g(p, n, b, q)$
(Prajapat et al. [14]);

(2) $C_\gamma^q\left(z^p + \sum_{k=n+p}^{\infty} \left(\frac{k+\mu}{p+\mu}\right)^r z^k; n+p, 0, p, \lambda, 1, b\right) = S_{n,q}^p(\mu, r, \gamma, b)$
($\mu \geq 0$ and $r \in \mathbb{N}_0$) (Srivastava et al. [18]);

(3) $C_0^q\left(z^p + \sum_{k=n+p}^{\infty} \left[1 + \frac{\zeta(k-p)}{p+r}\right]^\eta z^k; n+p, 0, p, \lambda, 1, b\right) = H_{n,q}^{p,r}(\zeta, \eta)$
($\zeta, \eta, r \in \mathbb{R}; \zeta \geq 0, \eta \geq 0, r \geq 0$) (Mahzoon and Latha [12]);

(4) $C_\gamma^q\left(\frac{z^p}{1-z}; n+p, 0, p, \lambda, \beta, b\right) = S_{n,p}^q(\gamma, \beta, b)$
(Altıntaş et al. [2]);

(5) $C_0^q\left(z^p + \sum_{k=n+p}^{\infty} \binom{\mu+k-1}{k-p} z^k; n+p, 0, p, \lambda, 1, b\right) = H_{n,q}^p(\mu, b)$
($\mu \geq 0$) (Raina and Srivastava [15]);

(6) $C_\gamma^q\left(\frac{z^p}{1-z}; n+p, 0, p, \lambda, p-q-\alpha, 1\right)$
 $= C_\gamma^q\left(\frac{z^p}{1-z}; n+p, 0, p, \lambda, 1, p-q-\alpha\right) = T_n(p, q, \alpha, \gamma)$
($0 \leq \alpha < p-q$) (Altıntaş [1]);

(7) $C_\gamma^q(g(z); n, 0, p, \lambda, \beta, b) = C_\gamma^q(g(z); n, p, \beta, b)$
(Srivastava and Orhan [17] and Aouf [4]);

(8) $C_0^0\left(\frac{z^p}{1-z}; n, m, p, \lambda, \beta, b\right) = T_{n-p}(m, p, \lambda, b, \beta)$
(El-Ashwah and Aouf [8]).

Also, we note that

$$\begin{aligned}
 (1) \quad & C_\gamma^q \left(z^p + \sum_{k=n}^{\infty} \left[\frac{p+\ell+\zeta(k-p)}{p+\ell} \right]^s z^k; n, 0, p, \lambda, \beta, b \right) \\
 &= C_\gamma^q(\zeta, \ell, s; n, p, \beta, b) \\
 &= \left\{ f \in T_p(n) : \left| \frac{1}{b} \left\{ \frac{z(I_p^s(\zeta, \ell)f(z))^{(q+1)} + \gamma z^2 (I_p^s(\zeta, \ell)f(z))^{(q+2)}}{(1-\gamma)(I_p^s(\zeta, \ell)f(z))^{(q)} + \gamma z (I_p^s(\zeta, \ell)f(z))^{(q+1)}} - (p-q) \right\} \right| < \beta, \right. \\
 &\quad \left. p, n \in \mathbb{N}; q, s \in \mathbb{N}_0; 0 \leq \gamma \leq 1; p > q; 0 < \beta \leq 1; \right. \\
 &\quad \left. \ell, \zeta \geq 0; b \in \mathbb{C}^*; z \in U \right\},
 \end{aligned}$$

where $I_p^s(\zeta, \ell)$ is an extended multiplier transformation (see Cătaş [7]), defined by

$$I_p^s(\zeta, \ell)f(z) = z^p - \sum_{k=n}^{\infty} \left[\frac{p+\ell+\zeta(k-p)}{p+\ell} \right]^s a_k z^k$$

($\ell, \zeta \geq 0; p \in \mathbb{N}$ and $s \in \mathbb{N}_0$);

$$\begin{aligned}
 (2) \quad & C_\gamma^q \left(\frac{z^p}{1-z}; n, m, p, \lambda, \beta, b \right) \\
 &= C_\gamma^q(n, m, p, \lambda, \beta, b) \\
 &= \left\{ f \in T_p(n) : \left| \frac{1}{b} \left\{ \frac{z(D_{\lambda,p}^m f(z))^{(q+1)} + \gamma z^2 (D_{\lambda,p}^m f(z))^{(q+2)}}{(1-\gamma)(D_{\lambda,p}^m f(z))^{(q)} + \gamma z (D_{\lambda,p}^m f(z))^{(q+1)}} - (p-q) \right\} \right| < \beta, \right. \\
 &\quad \left. p, n \in \mathbb{N}; q, m \in \mathbb{N}_0; b \in \mathbb{C}^*; 0 \leq \gamma \leq 1; p > q; \right. \\
 &\quad \left. 0 < \beta \leq 1; \lambda \geq 0 \right\}.
 \end{aligned}$$

Also let $R_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ denote the subclass $T_p(n)$ consisting of functions $f(z)$ of the form (1.9) and the function $g(z)$ of the form (1.10) which satisfy the following inequality:

$$(1.12) \quad \left| \frac{1}{b} \left\{ (1-\gamma) \frac{(D_{\lambda,p}^m (f*g)(z))^{(q)}}{z^{p-q}} + \gamma \frac{(D_{\lambda,p}^m (f*g)(z))^{(q+1)}}{(p-q)z^{p-q-1}} - \delta(p, q) \right\} \right| < \beta$$

($\lambda \geq 0; p, n \in \mathbb{N}; q, m \in \mathbb{N}_0; 0 \leq \gamma \leq 1; p > q; 0 < \beta \leq 1; b \in \mathbb{C}^*; z \in U$).

In this paper we shall study some properties of the classes $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ and $R_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ and derive several results for functions in the subclass $H_\gamma^q(g(z); n, m, p, \lambda, \beta, b, \alpha)$ of the function class $T_p(n)$, which is defined as follows:

A function $f(z) \in T_p(n)$ is said to belong to the class $H_\gamma^q(g(z); n, m, p, \lambda, \beta, b, \alpha)$ if $w = f(z)$ satisfies the following non-homogenous Cauchy–Euler

differential equation:

$$(1.13) \quad z^2 \frac{d^{q+2}w}{dz^{q+2}} + 2(1 + \alpha)z \frac{d^{q+1}w}{dz^{q+1}} + \alpha(1 + \alpha) \frac{d^q w}{dz^q} \\ = (p - q + \alpha)(p - q + \alpha + 1) \frac{d^q k}{dz^q},$$

where $k(z) \in C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ and $\alpha > q - p$, $\alpha \in R$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$.

2. Basic properties of the classes $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ and $R_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$. We begin by proving a necessary and sufficient condition for a function belonging to the class $T_p(n)$ to be in the class $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$.

Theorem 1. *Let the function $f(z) \in T_p(n)$ be defined by (1.9) and let $g(z)$ be defined by (1.10). Then $f(z)$ is in the class $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ if and only if*

$$(2.1) \quad \sum_{k=n}^{\infty} [k - p + \beta |b|] [1 + \gamma(k - q - 1)] \left[\frac{p + \lambda(k - p)}{p} \right]^m \delta(k, q) a_k b_k \\ \leq \beta |b| [1 + \gamma(p - q - 1)] \delta(p, q).$$

Proof. If the condition (2.1) holds true, we find from (1.9), (1.10) and (2.1) that

$$\left| z(D_{\lambda, p}^m(f * g)(z))^{(q+1)} + \gamma z^2(D_{\lambda, p}^m(f * g)(z))^{(q+2)} \right. \\ \left. - (p - q) \left[(1 - \gamma)(D_{\lambda, p}^m(f * g)(z))^{(q)} + \gamma z(D_{\lambda, p}^m(f * g)(z))^{(q+1)} \right] \right| \\ \left. - \beta \left| b \left[(1 - \gamma)(D_{\lambda, p}^m(f * g)(z))^{(q)} + \gamma z(D_{\lambda, p}^m(f * g)(z))^{(q+1)} \right] \right| \right| \\ = \left| \delta(p, q + 1) z^{p-q} - \sum_{k=n}^{\infty} \left[\frac{p + \lambda(k - p)}{p} \right]^m \delta(k, q + 1) a_k b_k z^{k-q} \right. \\ \left. + \gamma \delta(p, q + 2) z^{p-q} - \sum_{k=n}^{\infty} \gamma \left[\frac{p + \lambda(k - p)}{p} \right]^m \delta(k, q + 2) a_k b_k z^{k-q} \right. \\ \left. - (p - q) \left[(1 - \gamma) \delta(p, q) z^{p-q} - \sum_{k=n}^{\infty} (1 - \gamma) \left[\frac{p + \lambda(k - p)}{p} \right]^m \delta(k, q) a_k b_k z^{k-q} \right. \right. \\ \left. \left. + \gamma \delta(p, q + 1) z^{p-q} - \sum_{k=n}^{\infty} \gamma \left[\frac{p + \lambda(k - p)}{p} \right]^m \delta(k, q + 1) a_k b_k z^{k-q} \right] \right| \\ \left. - \beta \left| b \left[(1 - \gamma) \delta(p, q) z^{p-q} - \sum_{k=n}^{\infty} (1 - \gamma) \left[\frac{p + \lambda(k - p)}{p} \right]^m \delta(k, q) a_k b_k z^{k-q} \right. \right. \right. \\ \left. \left. + \gamma \delta(p, q + 1) z^{p-q} - \sum_{k=n}^{\infty} \gamma \left[\frac{p + \lambda(k - p)}{p} \right]^m \delta(k, q + 1) a_k b_k z^{k-q} \right] \right| \right|$$

$$\begin{aligned}
&= \left| \sum_{k=n}^{\infty} (k-p) [1 + \gamma(k-q-1)] \left[\frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k z^{k-q} \right. \\
&\quad \left. - \beta |b| \left[(1 + \gamma(p-q-1)) \delta(p, q) z^{p-q} \right. \right. \\
&\quad \left. \left. - \sum_{k=n}^{\infty} (1 + \gamma(k-q-1)) \left[\frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k z^{k-q} \right] \right| \\
&\leq \sum_{k=n}^{\infty} (k-p) [1 + \gamma(k-q-1)] \left[\frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k |z|^{k-p} \\
&\quad - \beta |b| \left\{ [1 + \gamma(p-q-1)] \delta(p, q) \right. \\
&\quad \left. - \sum_{k=n}^{\infty} [1 + \gamma(k-q-1)] \left[\frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k |z|^{k-p} \right\} \\
&\leq \sum_{k=n}^{\infty} [k-p + \beta |b|] [1 + \gamma(k-q-1)] \left[\frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k \\
&\quad - \beta |b| [1 + \gamma(p-q-1)] \delta(p, q) \leq 0
\end{aligned}$$

($z \in \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}$). Hence, by the maximum modulus theorem, $f(z) \in C_{\gamma}^q(g(z); n, p, \beta, b)$.

Conversely, let $f(z) \in C_{\gamma}^q(g(z); n, p, \beta, b)$ be given by (1.9) and $g(z)$ be given by (1.10). Then from (1.7) and (1.11), we have

$$\begin{aligned}
&\left| \frac{1}{b} \left\{ \frac{z(D_{\lambda, p}^m(f * g)(z))^{(q+1)} + \gamma z^2 (D_{\lambda, p}^m(f * g)(z))^{(q+2)}}{(1-\gamma)(D_{\lambda, p}^m(f * g)(z))^{(q)} + \gamma z (D_{\lambda, p}^m(f * g)(z))^{(q+1)}} - (p-q) \right\} \right| \\
(2.2) \quad &= \left| \frac{1}{b} \left\{ \frac{\sum_{k=n}^{\infty} (k-p) [1 + \gamma(k-q-1)] \left[\frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k z^{k-p}}{[1 + \gamma(p-q-1)] \delta(p, q) - \sum_{k=n}^{\infty} [1 + \gamma(k-q-1)] \left[\frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k z^{k-p}} \right\} \right| \\
&< \beta.
\end{aligned}$$

Putting $z = r$ ($0 \leq r < 1$) on the right-hand side of (2.2) and noting the fact that for $r = 0$, the resulting expression in the denominator is positive and remains so for all $r \in (0, 1)$, the desired inequality (2.1) follows upon letting $r \rightarrow 1^-$. \square

Theorem 2. *Let the function $f(z) \in T_p(n)$ be defined by (1.9) and $g(z)$ be defined by (1.10). Then $f(z)$ is in the class $R_{\gamma}^q(g(z); n, m, p, \lambda, \beta, b)$ if and only if*

$$(2.3) \quad \sum_{k=n}^{\infty} [p-q + \gamma(k-p)] \left[\frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k \leq \beta |b| (p-q).$$

Corollary 1. *Let the function $f(z) \in T_p(n)$ be given by (1.9) and $g(z)$ be defined by (1.10). If $f(z) \in C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$, then*

$$(2.4) \quad a_k \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{[k - p + \beta |b|] [1 + \gamma(k - q - 1)] \left[\frac{p + \lambda(k - p)}{p} \right]^m \delta(k, q) b_k}$$

($k \geq n$; $\lambda \geq 0$; $0 \leq \gamma \leq 1$; $0 < \beta \leq 1$; $b \in \mathbb{C}^*$; $p, n \in \mathbb{N}$; $q, m \in \mathbb{N}_0$).

The result is sharp for the function $f(z)$ given by

$$(2.5) \quad f(z) = z^p - \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{\delta(k, q) [k - p + \beta |b|] [1 + \gamma(k - q - 1)] \left[\frac{p + \lambda(k - p)}{p} \right]^m b_k} z^k$$

($k \geq n$; $\lambda \geq 0$; $0 \leq \gamma \leq 1$; $0 < \beta \leq 1$; $b \in \mathbb{C}^*$; $p, n \in \mathbb{N}$; $q, m \in \mathbb{N}_0$).

We next prove the following growth and distortion property for the functions of the form (1.9) belonging to the class $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$.

Theorem 3. *If a function $f(z)$ defined by (1.9) is in the class $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ and $g(z)$ defined by (1.10). Then*

$$(2.6) \quad \begin{aligned} & \left| |f(z)| - |z|^p \right| \\ & \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m \delta(n, q) b_n} |z|^n \end{aligned}$$

($\lambda \geq 0$; $p, n \in \mathbb{N}$; $q, m \in \mathbb{N}_0$; $0 \leq \gamma \leq 1$; $n > p > q$; $0 < \beta \leq 1$; $b \in \mathbb{C}^*$; $z \in U$) and (in general)

$$(2.7) \quad \begin{aligned} & \left| \left| f^{(r)}(z) \right| - \delta(p, r) |z|^{p-r} \right| \\ & \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] (n - q)! \delta(p, q)}{(n - p + \beta |b|) (n - r)! [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m b_n} |z|^{n-r} \end{aligned}$$

($z \in U$; $p, n \in \mathbb{N}$; $n > p$; $m, q \in \mathbb{N}_0$; $r \leq q < p$; $p > \max(r, q)$; $\lambda \geq 0$). The result is sharp for the function $f(z)$ given by

$$(2.8) \quad f(z) = z^p - \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m \delta(n, q) b_n} z^n$$

($n > p$; $p, n \in \mathbb{N}$).

Proof. In view of Theorem 1, we have

$$\begin{aligned} & (n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m \delta(n, q) b_n \sum_{k=n}^{\infty} a_k \\ & \leq \sum_{k=n}^{\infty} [k - p + \beta |b|] [1 + \gamma(k - q - 1)] \left[\frac{p + \lambda(k - p)}{p} \right]^m \delta(k, q) a_k b_k \\ & \leq \beta |b| [1 + \gamma(p - q - 1)] \delta(p, q), \end{aligned}$$

which readily yields

$$(2.9) \quad \sum_{k=n}^{\infty} a_k \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m \delta(n, q) b_n}.$$

Also, (2.1) yields

$$(2.10) \quad \sum_{k=n}^{\infty} k! a_k \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] (n - q)! \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m b_n}.$$

Now, by differentiating r times both sides of (1.9), we have

$$(2.11) \quad f^{(r)}(z) = \delta(p, r) z^{p-r} - \sum_{k=n}^{\infty} \delta(k, r) a_k z^{k-r}$$

($p, n \in \mathbb{N}$; $r \in \mathbb{N}_0$; $p > r$).

Theorem 3 follows from (2.9), (2.10) and (2.11). Finally, it is easy to see that the bounds in Theorem 1 are attained for the function $f(z)$ given by (2.8). \square

3. Properties of the class $H_{\gamma}^q(g(z); n, m, p, \lambda, \beta, b, \alpha)$. Applying the results of Section 2, which are obtained for the function $f(z)$ of the form (1.9) belonging to the class $C_{\gamma}^q(g(z); n, m, p, \lambda, \beta, b)$, we now derive the corresponding results for the function $f(z)$ belonging to the class $H_{\gamma}^q(g(z); n, m, p, \lambda, \beta, b, \alpha)$.

Theorem 4. *If a function $f(z)$ is defined by (1.9) and $g(z)$ is defined by (1.10), and $f(z)$ is in the class $H_{\gamma}^q(g(z); n, m, p, \lambda, \beta, b, \alpha)$. Then*

$$(3.1) \quad \begin{aligned} & \left| |f(z)| - |z|^p \right| \\ & \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] (p - q + \alpha)(p - q + \alpha + 1) \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m (n - q + \alpha) \delta(n, q) b_n} |z|^n \end{aligned}$$

and (in general)

$$(3.2) \quad \begin{aligned} & \left| \left| f^{(r)}(z) \right| - \delta(p, r) |z|^{p-r} \right| \\ & \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] (p - q + \alpha)(p - q + \alpha + 1)(n - q)! \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m (n - q + \alpha)(n - r)! b_n} |z|^{n-r} \end{aligned}$$

($p, n \in \mathbb{N}$; $m, q \in \mathbb{N}_0$; $r \leq q < p$; $p > \max(r, q)$; $0 \leq \gamma \leq 1$; $0 < \beta \leq 1$; $b \in \mathbb{C}^*$; $\lambda \geq 0$; $z \in U$). The results in (3.1) and (3.2) are sharp for the function $f(z)$ given by

$$(3.3) \quad f(z) = z^p - \frac{\beta |b| \delta(p, q) [1 + \gamma(p - q - 1)] (p - q + \alpha)(p - q + \alpha + 1)}{(n + \beta |b|) \delta(n + p, q) [1 + \gamma(n + p - q - 1)] (n + p - q + \alpha) b_{n+p}} z^n.$$

Proof. Assume that $f(z) \in T_p(n)$ is given by (1.9) and $g(z)$ given by (1.10). Also, let function $k(z) \in C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$, occurring in the non-homogenous differential equation (1.13) be of the form:

$$(3.4) \quad k(z) = z^p - \sum_{k=n}^{\infty} c_k z^k$$

($c_k \geq 0$; $n > p$; $p, n \in \mathbb{N}$). Then, we readily find from (1.13) that

$$(3.5) \quad a_k = \frac{(p - q + \alpha)(p - q + \alpha + 1)}{(k - q + \alpha)(k - q + \alpha + 1)} c_k$$

($k \geq n; p, n \in \mathbb{N}$), so that

$$(3.6) \quad f(z) = z^p - \sum_{k=n}^{\infty} a_k z^k = z^p - \sum_{k=n}^{\infty} \frac{(p - q + \alpha)(p - q + \alpha + 1)}{(k - q + \alpha)(k - q + \alpha + 1)} c_k z^k$$

($z \in U$), and

$$(3.7) \quad \|f(z) - |z|^p\| \leq |z|^n \sum_{k=n}^{\infty} \frac{(p - q + \alpha)(p - q + \alpha + 1)}{(k - q + \alpha)(k - q + \alpha + 1)} c_k$$

($z \in U$). Next, since $k(z) \in C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$, therefore, on using the assertion (2.4) of Corollary 1, we get the following coefficient inequality:

$$(3.8) \quad c_k \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m \delta(n, q) b_n}$$

($k \geq n$; $n > p > q$; $\lambda \geq 0$; $0 \leq \gamma \leq 1$; $0 < \beta \leq 1$; $p, n \in \mathbb{N}$; $q, m \in \mathbb{N}_0$; $b \in \mathbb{C}^*$), which in conjunction with (3.6) and (3.7) yields

$$(3.9) \quad \begin{aligned} & \|f(z) - |z|^p\| \\ & \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] (p - q + \alpha)(p - q + \alpha + 1) \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m \delta(n, q) b_n} |z|^n \\ & \times \sum_{k=n}^{\infty} \frac{1}{(k - q + \alpha)(k - q + \alpha + 1)} \end{aligned}$$

($z \in U$). Note that the following summation result holds

$$(3.10) \quad \begin{aligned} \sum_{k=n}^{\infty} \frac{1}{(k - q + \alpha)(k - q + \alpha + 1)} &= \sum_{k=n}^{\infty} \left(\frac{1}{(k - q + \alpha)} - \frac{1}{(k - q + \alpha + 1)} \right) \\ &= \frac{1}{(n - q + \alpha)}, \end{aligned}$$

where $\alpha \in \mathbb{R}^* = \mathbb{R} \setminus \{-n, -n - 1, \dots\}$. The assertion (3.1) of Theorem 4 follows from (3.9) and (3.10), respectively. The assertion (3.2) of Theorem 4

can be established similarly by applying (2.10), (2.11), (3.5) and (3.10), respectively. \square

4. Inclusion relations involving (n, θ) -neighborhood for the classes $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$, $R_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ and $H_\gamma^q(g(z); n, m, p, \lambda, \beta, b, \alpha)$. Following the works of Goodman [10], Ruscheweyh [16] and Altıntaş [1] (see also [2], [3] and [9]), we define the (n, θ) -neighborhood of a function $f^{(q)}(z)$ when $f \in T_p(n)$ by

$$(4.1) \quad N_{n,p}^\theta(f^{(q)}, k^{(q)}) = \left\{ k \in T_p(n) : k(z) = z^p - \sum_{k=n}^{\infty} c_k z^k \text{ and } \sum_{k=n}^{\infty} k \delta(k, q) |a_k - c_k| \leq \theta \right\}.$$

It follows from (4.1) that, if

$$(4.2) \quad h(z) = z^p$$

($p \in \mathbb{N}$), then

$$(4.3) \quad N_{n,p}^\theta(h^{(q)}) = \left\{ k \in T_p(n) : k(z) = z^p - \sum_{k=n}^{\infty} c_k z^k \text{ and } \sum_{k=n+p}^{\infty} k \delta(k, q) |c_k| \leq \theta \right\}.$$

Next, we establish inclusion relationships for the function classes $C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ and $R_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$, involving the (n, θ) -neighborhood $N_{n,p}^\theta(h^{(q)})$ defined by (4.3).

Theorem 5. *If $b_k \geq b_n$ ($k \geq n$) and*

$$(4.4) \quad \theta = \frac{n\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m b_n}$$

($p > |b|$), then

$$(4.5) \quad C_\gamma^q(g(z); n, m, p, \lambda, \beta, b) \subset N_{n,p}^\theta(h^{(q)}).$$

Proof. Let $f \in C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$. Then, in view of the assertion (2.1) of Theorem 1, and the given condition that $b_k \geq b_n$ ($k \geq n$), we have

$$(4.6) \quad (n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m b_n \sum_{k=n}^{\infty} \delta(k, q) a_k \leq \beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)$$

so that

$$(4.7) \quad \sum_{k=n}^{\infty} \delta(k, q) a_k \leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p + \lambda(n - p)}{p} \right]^m b_n}.$$

On the other hand, we also find from (2.1) and (4.7) that

$$\begin{aligned} \sum_{k=n}^{\infty} k\delta(k, q)a_k &\leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{[1 + \gamma(n - q - 1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n} + (p - \beta |b|) \sum_{k=n}^{\infty} \delta(k, q)a_k \\ &\leq \frac{\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{[1 + \gamma(n - q - 1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n} \\ &\quad + \frac{(p - \beta |b|)\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n}, \end{aligned}$$

that is

$$(4.8) \quad \sum_{k=n}^{\infty} \delta(k, q)ka_k \leq \frac{n\beta |b| \delta(p, q) [1 + \gamma(p - q - 1)]}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n} = \theta.$$

This evidently completes the proof of Theorem 5. □

Remark 1. (i) Taking $g(z) = \frac{z^p}{1-z}$, $b = \gamma$, $m = 0$ and $\gamma = \lambda$ in Theorem 5, we obtain the result obtained by Altıntaş et al. [2, Theorem 2];

(ii) Taking $g(z) = \frac{z^p}{1-z}$, $b = 1$, $\beta = p - \alpha$ ($0 \leq \alpha < p$) and $\gamma = \lambda$ in Theorem 5, we obtain the result obtained by Altıntaş [1, Theorem 2].

Putting $g(z) = z^p + \sum_{k=n}^{\infty} \left[\frac{p+\ell+\zeta(k-p)}{p+\ell} \right]^s z^k$ ($\ell, \zeta \geq 0$; $s \in \mathbb{N}_0$) and $m = 0$ in Theorem 5, we obtain the following corollary.

Corollary 2. *If $f(z) \in T_p(n)$ is in the class $C_\gamma^q(\zeta, \ell, s; n, p, \beta, b)$, then*

$$C_\gamma^q(\zeta, \ell, s; n, p, \beta, b) \subset N_{n,p}^\theta(h^{(q)}),$$

where $h(z)$ is given by (4.2) and

$$\theta = \frac{n\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)]} \left(\frac{p + \ell}{p + \ell + \zeta(n - p)} \right)^s.$$

Putting $g(z) = z^p + \sum_{k=n}^{\infty} \left[\frac{p+\zeta(k-p)}{p} \right]^s z^k$ ($\zeta \geq 0$; $s \in \mathbb{N}_0$) and $m = 0$ in Theorem 5, we obtain the following corollary.

Corollary 3. *If $f(z) \in T_p(n)$ is in the class $C_\gamma^q(\zeta, s; n, p, \beta, b)$, then*

$$C_\gamma^q(\zeta, s; n, p, \beta, b) \subset N_{n,p}^\theta(h^{(q)}),$$

where $h(z)$ is given by (4.2) and

$$\theta = \frac{n\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)]} \left(\frac{p}{p + \zeta(n - p)} \right)^s.$$

Theorem 6. *If*

$$(4.9) \quad \theta = \frac{n\beta |b| (p-q)}{[p-q + \gamma(n-p)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n},$$

then

$$(4.10) \quad R_\gamma^q(g(z); n, m, p, \lambda, \beta, b) \subset N_{n,p}^\theta(h^{(q)}).$$

Proof. Let $f \in R_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$. Then, in view of the assertion (2.3) of Theorem 2, we have

$$\begin{aligned} & \frac{[p-q + \gamma(n-p)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n}{n} \sum_{k=n}^{\infty} \delta(k, q) k a_k \\ & \leq \sum_{k=n}^{\infty} [p-q + \gamma(k-p)] \left[\frac{p+\lambda(k-p)}{p} \right]^m \delta(k, q) a_k b_k \\ & \leq \beta |b| (p-q), \end{aligned}$$

so that

$$(4.11) \quad \sum_{k=n}^{\infty} \delta(k, q) k a_k \leq \frac{n\beta |b| (p-q)}{[p-q + \gamma(n-p)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n} = \theta,$$

which by means of the definition (4.1), establishes the inclusion (4.10) asserted by Theorem 6. \square

Theorem 7. *If $f(z) \in T_p(n)$ is in the class $H_\gamma^q(g(z); n, m, p, \lambda, \beta, b, \alpha)$, then*

$$(4.12) \quad H_\gamma^q(g(z); n, m, p, \lambda, \beta, b, \alpha) \subset N_{n,p}^\theta(f^{(q)}, k^{(q)}),$$

where $k(z)$ is given by (1.13) and

$$(4.13) \quad \theta = \frac{n\beta |b| [1 + \gamma(p-q-1)] [n + (p-q + \alpha)(p-q + \alpha + 2)] \delta(p, q)}{(n-p + \beta |b|) [1 + \gamma(n-q-1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m (n-q + \alpha) b_n}.$$

Proof. Suppose that $f(z) \in H_\gamma^q(g(z); n, m, p, \lambda, \beta, b, \alpha)$. Then upon substituting from (3.5) into the following coefficient inequality

$$(4.14) \quad \sum_{k=n}^{\infty} k \delta(k, q) |a_k - c_k| \leq \sum_{k=n}^{\infty} k \delta(k, q) |c_k| + \sum_{k=n}^{\infty} k \delta(k, q) |a_k|$$

($a_k; c_k \geq 0$), we readily obtain

$$(4.15) \quad \sum_{k=n}^{\infty} k\delta(k, q) |a_k - c_k| \leq \sum_{k=n}^{\infty} k\delta(k, q) |c_k| + \sum_{k=n}^{\infty} k\delta(k, q) \frac{(p-q+\alpha)(p-q+\alpha+1)}{(k-q+\alpha)(k-q+\alpha+1)} |c_k|.$$

Now, since $k(z) \in C_{\gamma}^q(g(z); n, m, p, \lambda, \beta, b)$ the second assertion (4.8) yields

$$(4.16) \quad k\delta(k, q)c_k \leq \frac{n\beta |b| \delta(p, q) [1 + \gamma(p - q - 1)]}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n}.$$

Finally, by making use of (4.8) as well as (4.16) on the right-hand side of (4.15), we find that

$$\begin{aligned} & \sum_{k=n+p}^{\infty} \delta(k, q)k |a_k - c_k| \\ & \leq \frac{n\beta |b| [1 + \gamma(p - q - 1)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n} \\ & \quad \times \left(1 + \sum_{k=n+p}^{\infty} \frac{(p - q + \alpha)(p - q + \alpha + 1)}{(k - q + \alpha)(k - q + \alpha + 1)} \right) \\ & = \frac{n\beta |b| [1 + \gamma(p - q - 1)] [n + (p - q + \alpha)(p - q + \alpha + 2)] \delta(p, q)}{(n - p + \beta |b|) [1 + \gamma(n - q - 1)] (n + p - q + \alpha) \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n} \\ & = \theta, \end{aligned}$$

we conclude that $f \in N_{n,p}^{\theta}(f^{(q)}, k^{(q)})$. This evidently completes the proof of Theorem 7. \square

5. Neighborhood for the classes $C_{\gamma}^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b)$ and $R_{\gamma}^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b)$. In this section we determine the neighborhood for the classes $C_{\gamma}^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b)$ and $R_{\gamma}^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b)$ which we define as follows. A function $f \in T_p(n)$ is said to be in the class $C_{\gamma}^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b)$ if there exists a function $k \in C_{\gamma}^q(g(z); n, m, p, \lambda, \beta, b)$ such that

$$(5.1) \quad \left| \frac{f(z)}{k(z)} - 1 \right| < p - \zeta$$

($z \in U; 0 \leq \zeta < p$).

Theorem 8. If $k(z) \in C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$ and

$$\zeta = p - \frac{\theta(n-p+\beta|b|)[1+\gamma(n-q-1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n}{n \left\{ (n-p+\beta|b|)[1+\gamma(n-q-1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m \delta(n, q) b_n - \beta|b|[1+\gamma(p-q-1)]\delta(p, q) \right\}},$$

then

$$(5.2) \quad N_{n,p}^\theta(k^{(q)}) \subset C_\gamma^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b),$$

where

$$\begin{aligned} \theta \leq np & \left[\delta(n, q) - \beta|b|[1+\gamma(p-q-1)]\delta(p, q) \right. \\ & \left. \times \left\{ (n-p+\beta|b|)[1+\gamma(n-q-1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n \right\}^{-1} \right]. \end{aligned}$$

Proof. Suppose that $f \in N_{n,p}^\theta(k^{(q)})$, then we find from the definition (4.1) that

$$(5.3) \quad \sum_{k=n}^{\infty} \delta(k, q) k |a_k - c_k| \leq \theta,$$

which implies the coefficient inequality

$$(5.4) \quad \sum_{k=n}^{\infty} |a_k - c_k| \leq \frac{\theta}{n\delta(n, q)}$$

($p > q$; $n, p \in \mathbb{N}$, $q \in \mathbb{N}_0$). Next, since $k(z) \in C_\gamma^q(g(z); n, m, p, \lambda, \beta, b)$, we have

$$\sum_{k=n}^{\infty} c_k \leq \frac{\beta|b|[1+\gamma(p-q-1)]\delta(p, q)}{(n-p+\beta|b|)[1+\gamma(n-q-1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m \delta(n, q) b_n},$$

so that

$$\begin{aligned} \left| \frac{f(z)}{k(z)} - 1 \right| & \leq \frac{\sum_{k=n}^{\infty} |a_k - c_k|}{1 - \sum_{k=n}^{\infty} |c_k|} \\ & \leq \frac{\frac{\theta}{n\delta(n, q)}}{1 - \frac{\beta|b|[1+\gamma(p-q-1)]\delta(p, q)}{(n-p+\beta|b|)[1+\gamma(n-q-1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m \delta(n, q) b_n}} \\ & = \frac{\theta(n-p+\beta|b|)[1+\gamma(n-q-1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n}{n \left\{ (n-p+\beta|b|)[1+\gamma(n-q-1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m \delta(n, q) b_n - \beta|b|\delta(p, q)[1+\gamma(p-q-1)] \right\}} \\ & = p - \zeta, \end{aligned}$$

because by the assumption

$$\zeta = p - \frac{\theta(n-p+\beta|b|)[1+\gamma(n-q-1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n}{n \left\{ (n-p+\beta|b|)[1+\gamma(n-q-1)] \left[\frac{p+\lambda(n-p)}{p} \right]^m \delta(n,q)b_n - \beta|b|\delta(p,q)[1+\gamma(p-q-1)] \right\}}.$$

This implies that $f \in C_{\gamma}^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b)$. \square

Similarly, we can prove the following theorem.

Theorem 9. *If $k(z) \in R_{\gamma}^q(g(z); n, m, p, \lambda, \beta, b)$ and*

$$(5.5) \quad \zeta = p - \frac{\theta[p-q+\gamma(n-p)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n}{n \left\{ [p-q+\gamma(n-p)] \left[\frac{p+\lambda(n-p)}{p} \right]^m \delta(n,q)b_n - \beta|b|(p-q) \right\}},$$

then

$$(5.6) \quad N_{n,p}^{\theta}(k^{(q)}) \subset R_{\gamma}^{q,\zeta}(g(z); n, m, p, \lambda, \beta, b),$$

where

$$\theta \leq np \left[\delta(n,q) - \beta|b|(p-q) \left\{ [p-q+\gamma(n-p)] \left[\frac{p+\lambda(n-p)}{p} \right]^m b_n \right\}^{-1} \right].$$

REFERENCES

- [1] Altıntaş, O., *Neighborhoods of certain p -valently analytic functions with negative coefficients*, Appl. Math. Comput. **187** (2007), 47–53.
- [2] Altıntaş, O., Irmak, H. and Srivastava, H. M., *Neighborhoods for certain subclasses of multivalently analytic functions defined by using a differential operator*, Comput. Math. Appl. **55** (2008), 331–338.
- [3] Altıntaş, O., Özkan, Ö. and Srivastava, H. M., *Neighborhoods of a certain family of multivalent functions with negative coefficient*, Comput. Math. Appl. **47** (2004), 1667–1672.
- [4] Aouf, M. K., *Inclusion and neighborhood properties for certain subclasses of analytic functions associated with convolution structure*, J. Austral. Math. Anal. Appl. **6**, no. 2 (2009), Art. 4, 1–10.
- [5] Aouf, M. K., Mostafa, A. O., *On a subclass of n - p -valent prestarlike functions*, Comput. Math. Appl. **55** (2008), 851–861.
- [6] Aouf, M. K., Seoudy, T. M., *On differential sandwich theorems of analytic functions defined by certain linear operator*, Ann. Univ. Marie Curie-Skłodowska Sect. A, **64** (2) (2010), 1–14.
- [7] Cătaş, A., *On certain classes of p -valent functions defined by multiplier transformations*, Proceedings of the International Symposium on Geometric Function Theory and Applications: GFTA 2007 Proceedings (İstanbul, Turkey; 20-24 August 2007) (S. Owa and Y. Polatoglu, Editors), pp. 241–250, TC İstanbul Kültür University Publications, Vol. 91, TC İstanbul Kültür University, İstanbul, Turkey, 2008.
- [8] El-Ashwah, R. M., Aouf, M. K., *Inclusion and neighborhood properties of some analytic p -valent functions*, General Math. **18**, no. 2 (2010), 173–184.
- [9] Frasin, B. A., *Neighborhoods of certain multivalent analytic functions with negative coefficients*, Appl. Math. Comput. **193**, no. 1 (2007), 1–6.

-
- [10] Goodman, A. W., *Univalent functions and non-analytic curves*, Proc. Amer. Math. Soc. **8** (1957), 598–601.
- [11] Kamali, M., Orhan, H., *On a subclass of certain starlike functions with negative coefficients*, Bull. Korean Math. Soc. **41**, no. 1 (2004), 53–71.
- [12] Mahzoon, H., Latha, S., *Neighborhoods of multivalent functions*, Internat. J. Math. Analysis, **3**, no. 30 (2009), 1501–1507.
- [13] Orhan, H., Kiziltunc, H., *A generalization on subfamily of p -valent functions with negative coefficients*, Appl. Math. Comput., **155** (1004), 521–530.
- [14] Prajapat, J. K., Raina, R. K. and Srivastava, H. M., *Inclusion and neighborhood properties of certain classes of multivalently analytic functions associated with convolution structure*, JIPAM. J. Inequal. Pure Appl. Math. **8**, no. 1 (2007), Article 7, 8 pp. (electronic).
- [15] Raina, R. K., Srivastava, H. M., *Inclusion and neighborhood properties of some analytic and multivalent functions*, J. Inequal. Pure Appl. Math. **7**, no. 1 (2006), 1–6.
- [16] Ruscheweyh, St., *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **81** (1981), 521–527.
- [17] Srivastava, H. M., Orhan, H., *Coefficient inequalities and inclusion relations for some families of analytic and multivalent functions*, Applied Math. Letters, **20**, no. 6 (2007), 686–691.
- [18] Srivastava, H. M., Suchithra, K., Stephen, B. A. and Sivasubramanian, S., *Inclusion and neighborhood properties of certain subclasses of analytic and multivalent functions of complex order*, JIPAM. J. Inequal. Pure Appl. Math. **7**, no. 5 (2006), Article 191, 8 pp. (electronic).

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