doi: 10.2478/v10062-011-0008-3

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXV, NO. 1, 2011 SECTIO A 97–108

ROSHAN LAL, SUSHEEL KUMAR[1] and SUNIL HANS[2] [3]

On the zeros of polynomials and analytic functions

Abstract. For a polynomial of degree n, we have obtained some results, which generalize and improve upon the earlier well known results (under certain conditions). A similar result is also obtained for analytic function.

1. Introduction and statement of results. The following theorem is due to Pellet ([6], [5, p. 128]).

Theorem A. *Let* $q(z) = a_0 + a_1 z + \ldots + a_p z^p + \ldots + a_n z^n$, $a_p \neq 0$, *be a polynomial of degree* n. *If the polynomial*

$$Q_p(z) = |a_0| + |a_1|z + \ldots + |a_{p-1}|z^{p-1} - |a_p|z^p + |a_{p+1}|z^{p+1} + \ldots + |a_n|z^n,$$

has two positive zeros r *and* R, $r < R$, *then* $q(z)$ *has exactly* p *zeros in the disc*

$$|z| \leq r$$

and no zero in the annular ring

$$r < |z| < R.$$

The next result is due to Jayal, Labelle and Rahman [4].

[1]The research of the author is supported by UGC, New Delhi; F. No. 17-52/98(SA-I).
[2]The research of the author is supported by CSIR, New Delhi; F. No. -9/466(95)/2007-EMR-I.
[3]Corresponding author.
2000 *Mathematics Subject Classification.* 30C15, 30A10.
Key words and phrases. Polynomial, analytic function, zeros.

Theorem B. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0.$$

Then $p(z)$ *has all its zeros in*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}. \tag{1}$$

Gardner and Govil [1] improved Theorem B as follows.

Theorem C. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0.$$

Then $p(z)$ *has all its zeros in the annular ring*

$$\frac{|a_0|}{a_n - a_0 + |a_n|} \leq |z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}. \tag{2}$$

Recently Jain [3] proved the following result for the upper bound involving coefficients of the polynomial.

Theorem D. *Let* $q(z) = a_0 + a_1 z + \ldots + a_{p-1} z^{p-1} + a_p z^p + \ldots + a_n z^n$, *be a polynomial of degree* n *such that* $a_p \neq a_{p-1}$ *for some* $p \in \{1, 2, \ldots, n\}$. *Set*

$$M = M_p := \sum_{j=p+1}^{n} |a_j - a_{j-1}| + |a_n| \quad (1 \leq p \leq n-1), \;\; M_n := |a_n|$$

and

$$m = m_p := \sum_{j=1}^{p-1} |a_j - a_{j-1}| \quad (2 \leq p \leq n), \;\; m_1 := 0.$$

Suppose that

$$\frac{p}{M} \frac{|a_p - a_{p-1}|}{p+1} < 1$$

and that

$$|a_0| + m \frac{p}{M} \frac{|a_p - a_{p-1}|}{p+1} < \left(\frac{p}{M}\right)^p \left(\frac{|a_p - a_{p-1}|}{p+1}\right)^{p+1}.$$

Then $q(z)$ *has at least* p *zeros in*

$$|z| < \frac{p}{M} \frac{|a_p - a_{p-1}|}{p+1}.$$

Jain [3] again, in the same paper proved the following.

Theorem E. *Let* $q(z) = a_0 + a_1 z + \ldots + a_{p-1} z^{p-1} + a_p z^p + \ldots + a_n z^n$, *be a polynomial of degree* n *such that* $a_p \neq a_{p-1}$ *for some* $p \in \{1, 2, \ldots, n-1\}$,

$$|\arg a_k - \beta| \leq \alpha \leq \frac{\pi}{2},$$

$k = 0, 1, 2 \ldots, n$, for some real β and α and

$$|a_n| \geq |a_{n-1}| \geq \ldots \geq |a_1| \geq |a_0|.$$

Let

$$L = L_p := |a_n| + (|a_n| - |a_p|)\cos\alpha + \sum_{j=p+1}^{n} (|a_j| + |a_{j-1}|)\sin\alpha$$

and

$$l = l_p := (|\alpha_{p-1}| - |a_0|)\cos\alpha + \sum_{j=1}^{p-1}(|a_j| + |a_{j-1}|)\sin\alpha \quad (2 \leq p \leq n-1), \; l_1 := 0.$$

Suppose that

$$|a_0| + l\frac{p}{L}\frac{|a_p - a_{p-1}|}{p+1} < \left(\frac{p}{L}\right)^p \left(\frac{|a_p - a_{p-1}|}{p+1}\right)^{p+1}.$$

Then $q(z)$ has at least p zeros in

$$|z| < \frac{p}{L}\frac{|a_p - a_{p-1}|}{p+1}.$$

In this paper, firstly we prove the following.

Theorem 1. *Let $q(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{p-1} z^{p-1} + a_p z^p + \ldots + a_n z^n$ be a polynomial of degree n such that $a_p \neq a_{p-1}$ for some $p \in \{1, 2, \ldots, n\}$, with coefficients a_j, $j = 0, 1, 2, \ldots, n$, satisfying*

$$a_n \geq a_{n-1} \geq \ldots \geq a_p > a_{p-1} \geq \ldots \geq a_1 \geq a_0 \tag{3}$$

and

$$\left(\frac{p}{M_1}\right)^p \left(\frac{a_p - a_{p-1}}{p+1}\right)^{p+1} > |a_0| + \frac{p}{M_1}\left(\frac{a_p - a_{p-1}}{p+1}\right)(a_{p-1} - a_0) \tag{4}$$

where $M_1 = a_n + |a_n| - a_p$.

Then $q(z)$ has at least p zeros in

$$\frac{|a_0|}{a_n - a_0 + \rho_1^n |a_n|} \leq |z| < \rho_1 = \frac{p}{(|a_n| - a_n + a_p)} \frac{(a_p - a_{p-1})}{(p+1)}. \tag{5}$$

Remark 1. In Theorem 1, we have

$$M_1 = |a_n| + a_n - a_p$$

for $1 \leq p \leq (n-1)$ and $M_1 = |a_n|$ for $p = n$. The value $M_1 = |a_n| + a_n - a_p$ serves the purpose for $1 \leq p \leq n$ (see also equality (1.6) of Jain [3]).

For the case $p = n$, in Theorem 1, we have the following.

Corollary 1. *Let $q(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a polynomial of degree n such that*

$$a_n > a_{n-1} \geq a_{n-2} \geq \ldots \geq a_1 \geq a_0, \tag{6}$$

and

$$\left(\frac{n}{|a_n|}\right)^n \left(\frac{a_n - a_{n-1}}{n+1}\right)^{n+1} > |a_0| + \frac{n}{n+1}\left(\frac{a_n - a_{n-1}}{|a_n|}\right)(a_{n-1} - a_0) \tag{7}$$

then $q(z)$ has all its zeros in

$$\frac{|a_0|}{a_n - a_0 + \rho_2^n |a_n|} \leq |z| < \rho_2 = \frac{n}{n+1} \frac{(a_n - a_{n-1})}{|a_n|}. \tag{8}$$

Remark 2. Corollary 1 is a refinement of Theorem B due to Joyal, Labelle and Rahman [4] as well as Theorem C due to Gardner and Govil [1] under the conditions (6) and (7).

As it can be shown easily from (8) and (2) that

$$\frac{n}{n+1}\frac{(a_n - a_{n-1})}{|a_n|} < \frac{a_n - a_0 + |a_0|}{|a_n|}$$

is always true.

And also

$$\frac{|a_0|}{a_n - a_0 + \rho_2^n |a_n|} > \frac{|a_0|}{a_n + |a_n| - a_0}$$

for

$$\rho_2 = \frac{n}{n+1}\frac{a_n - a_{n-1}}{|a_n|}.$$

Remark 3. If we take $a_0 > 0$, then Corollary 1 gives a refinement of a result due to Jain [3, Corollary 1].

Instead of proving Theorem 1, we prove the following result. Theorem 1 can be proved in a similar way as the next result (Theorem 2) except the only change that is in Theorem 1 $p \in \{1, 2, \ldots, n\}$.

Theorem 2. *Let $q(z) = a_0 + a_1 z + \ldots + a_{p-1} z^{p-1} + a_p z^p + \ldots a_n z^n$ be a polynomial of degree n such that $a_p \neq a_{p-1}$ for some $p \in \{1, \ldots, n-1\}$, with the coefficients a_j, $j = 0, 1, \ldots, n$, for some $K \geq 1$, satisfying*

$$K a_n \geq a_{n-1} \geq \ldots \geq a_p > a_{p-1} \geq \ldots \geq a_1 \geq a_0 \tag{9}$$

and

$$\left(\frac{p}{M_2}\right)^p \left(\frac{a_p - a_{p-1}}{p+1}\right)^{p+1} > |a_0| + \frac{p}{M_2}\left(\frac{a_p - a_{p-1}}{p+1}\right)(a_{p-1} - a_0), \tag{10}$$

where

$$M_2 = K(a_n + |a_n|) - a_p. \tag{11}$$

Then $q(z)$ has at least p zeros in

$$\frac{|a_0|}{Ka_n+(K-1)|a_n|-a_0+\rho_3^n|a_n|}\le|z|\le\rho_3=\frac{p}{M_2}\frac{(a_p-a_{p-1})}{p+1}\tag{12}$$

where we assume that $\rho_3<1$.

For the case $a_0>0$, we have the following.

Corollary 2. *Let $q(z)=a_0+a_1z+a_2z^2+\ldots+a_{p-1}z^{p-1}+a_pz^p+\ldots+a_nz^n$ be a polynomial of degree n, with the condition $a_p\ne a_{p-1}$ and for some $p\in\{1,2,\ldots,n-1\}$, $K\ge1$ satisfying*

$$Ka_n\ge a_{n-1}\ge\ldots\ge a_p>a_{p-1}\ge\ldots\ge a_0>0\tag{13}$$

and

$$\left(\frac{p}{M_3}\right)^p\left(\frac{a_p-a_{p-1}}{p+1}\right)^{p+1}>a_0+\frac{p}{M_3}\left(\frac{a_p-a_{p-1}}{p+1}\right)(a_{p-1}-a_0),\tag{14}$$

where

$$M_3=2Ka_n-a_p.\tag{15}$$

Then $q(z)$ has at least p zeros in

$$\frac{a_0}{(2K-1+\rho_4^n)a_n-a_0}\le|z|<\rho_4=\frac{p}{p+1}\frac{(a_p-a_{p-1})}{(2Ka_n-a_p)}.\tag{16}$$

Remark 4. As $\frac{a_p-a_{p-1}}{2Ka_n-a_p}<1$ (by (13)), we have $\frac{p}{p+1}\left(\frac{a_p-a_{p-1}}{2Ka_n-a_p}\right)=\rho_4<1$.

For the polynomials with complex coefficients, we have been able to prove the following.

Theorem 3. *Let $q(z)=a_0+a_1z+\ldots+a_{p-1}z^{p-1}+a_pz^p+\ldots+a_nz^n$ be a polynomial of degree n such that $a_p\ne a_{p-1}$ for some $p\in\{1,2,\ldots,n-1\}$, for some real β and α*

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2},\quad j=0,1,\ldots,n,$$

and for some $K\ge1$,

$$K|a_n|\ge|a_{n-1}|\ge\ldots\ge|a_1|\ge|a_0|\tag{17}$$

and

$$\left(\frac{p}{M_4}\right)^p\left(\frac{|a_p-a_{p-1}|}{p+1}\right)^{p+1}>|a_0|+\frac{p}{M_4}\frac{|a_p-a_{p-1}|}{(p+1)}m'\tag{18}$$

where

$$\begin{aligned}M_4&=K|a_n|+(K|a_n|-|a_p|)\cos\alpha+(K|a_n|+|a_{n-1}|)\sin\alpha\\&\quad+\sum_{j=p+1}^{n-1}(|a_j|+|a_{j-1}|)\sin\alpha,\end{aligned}\tag{19}$$

$$m' = (|a_{p-1}| - |a_0|)\cos\alpha + \sum_{j=1}^{p-1}(|a_j| + |a_{j-1}|)\sin\alpha. \tag{20}$$

Then $q(z)$ has at least p zeros in

$$|z| < \rho_5 = \frac{p}{p+1}\frac{(|a_p - a_{p-1}|)}{M_4}. \tag{21}$$

Remark 5. In the case $K = 1$, the above theorem reduces to Theorem E due to Jain [3].

Remark 6. $\rho_5 < 1$, as can be verified by using (19), (17) and Lemma 1.

Now we turn to the study of zeros of an analytic function. In this direction, we have been able to prove the following.

Theorem 4. *Let the function $f(z) = \sum_{j=0}^{\infty} a_j z^j$ ($\not\equiv 0$) be analytic in $|z| \leq \rho_6$, for some $p \in N$ such that $\dfrac{a_{p-1}}{a_p} < 2 + \dfrac{1}{p}$. Assume that*

$$a_0 \geq a_1 \geq \ldots \geq a_{p-1} > a_p \geq a_{p+1} \geq \ldots \tag{22}$$

and

$$\left(\frac{p}{a_p}\right)^p \left(\frac{a_{p-1} - a_p}{p+1}\right)^{p+1} > |a_0| + \frac{p}{a_p}\left(\frac{a_{p-1} - a_p}{p+1}\right)(a_0 - a_{p-1}). \tag{23}$$

Then the function $f(z)$ has at least p zeros in

$$|z| < \rho_6 = \frac{p}{p+1}\left(\frac{a_{p-1} - a_p}{a_p}\right). \tag{24}$$

2. Lemma. For the proof of the theorems, we need the following lemma.

Lemma 1. *If a_j is any complex number with*

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2},$$

for certain real β and α, then

$$|a_j - a_{j-1}| \leq ||a_j| - |a_{j-1}||\cos\alpha + (|a_j| + |a_{j-1}|)\sin\alpha.$$

This lemma is due to Govil and Rahman (proof of Theorem 2 of [2]).

3. Proofs of Theorems.

Proof of Theorem 2. Consider

$$
\begin{aligned}
g(z) &= (1-z)q(z)\\
&= (1-z)(a_0 + a_1 z + \ldots + a_{p-1}z^{p-1} + a_p z^p + \ldots + a_n z^n)\\
&= a_0 + \sum_{j=1}^{p-1}(a_j - a_{j-1})z^j + (a_p - a_{p-1})z^p\\
&\quad + \sum_{j=p+1}^{n}(a_j - a_{j-1})z^j - a_n z^{n+1}.\\
&= \phi(z) + \psi(z),
\end{aligned}
\tag{25}
$$

where

$$\phi(z) = a_0 + \sum_{j=1}^{p-1}(a_j - a_{j-1})z^j,$$

and

$$\psi(z) = (a_p - a_{p-1})z^p + \sum_{j=p+1}^{n}(a_j - a_{j-1})z^j - a_n z^{n+1}.$$

Now for $|z| = \rho_3$ ($\rho_3 < 1$ (as assumed)) and $p \le n-1$,

$$
\begin{aligned}
|\psi(z)| &\ge |a_p - a_{p-1}|\rho_3^p - \left\{ \sum_{j=p+1}^{n} |a_j - a_{j-1}|\rho_3^j + |a_n|\rho_3^{n+1} \right\}\\
&\ge (a_p - a_{p-1})\rho_3^p - \rho_3^{p+1}\Bigg\{ |a_n|\rho_3^{n-p} + |a_n - a_{n-1}|\rho_3^{n-p}\\
&\qquad + \sum_{j=p+1}^{n-1} |a_j - a_{j-1}|\rho_3^{j-(p+1)} \Bigg\}\\
&\ge (a_p - a_{p-1})\rho_3^p - \rho_3^{p+1}\Bigg\{ |a_n| + |Ka_n - a_{n-1}|\\
&\qquad + (K-1)|a_n| + \sum_{j=p+1}^{n-1} |a_j - a_{j-1}| \Bigg\}\\
&= (a_p - a_{p-1})\rho_3^p - \rho_3^{p+1}\Big\{ |a_n| + Ka_n - a_{n-1}\\
&\qquad + (K-1)|a_n| + a_{n-1} - a_p \Big\}\\
&= (a_p - a_{p-1})\rho_3^p - \rho_3^{p+1}\{K(a_n + |a_n|) - a_p\}\\
&= \left(\frac{p}{M_2}\right)^p \left(\frac{a_p - a_{p-1}}{p+1}\right)^{p+1}
\end{aligned}
$$

$$> |a_0| + \frac{p}{M_2}\left(\frac{a_p - a_{p-1}}{p+1}\right)(a_{p-1} - a_0) \quad \text{by (10).}$$
$$= |a_0| + \rho_3(a_{p-1} - a_0).$$

Thus for $|z| = \rho_3$,

$$|\psi(z)| > |a_0| + \rho_3(a_{p-1} - a_0). \tag{26}$$

On the other hand, for $|z| = \rho_3$,

$$\begin{aligned} |\phi(z)| &\leq |a_0| + \sum_{j=1}^{p-1} |a_j - a_{j-1}|\rho_3^j \\ &\leq |a_0| + \rho_3(a_{p-1} - a_0). \end{aligned} \tag{27}$$

From (26) and (27) it follows that $|\psi(z)| > |\phi(z)|$ for $|z| = \rho_3$. By Rouche's theorem, $g(z) = \phi(z) + \psi(z)$ and $\psi(z)$ has same number of zeros in $|z| < \rho_3$. But $\psi(z)$ has at least p zeros in $|z| < \rho_3$. Therefore $g(z)$ and hence $q(z)$ has at least p zeros in

$$|z| < \rho_3.$$

This proves one part of Theorem 2.

Now it remains to prove that there are no zeros of $q(z)$ in

$$|z| < \frac{|a_0|}{Ka_n + (K-1)|a_n| - a_0 + \rho_3^n |a_n|}.$$

Let

$$\begin{aligned} g(z) = (1-z)q(z) &= a_0 + \sum_{j=1}^{n}(a_j - a_{j-1})z^j - a_n z^{n+1} \\ &= a_0 + h(z). \end{aligned} \tag{28}$$

Now for $|z| = \rho_3$, $(\rho_3 < 1)$ we have

$$\begin{aligned} \max_{|z|=\rho_3} |h(z)| &\leq \sum_{j=1}^{n} |a_j - a_{j-1}|\rho_3^j + |a_n|\rho_3^{n+1} \\ &\leq \rho_3\left\{|a_n - a_{n-1}| + \sum_{j=1}^{n-1} |a_j - a_{j-1}| + |a_n|\rho_3^n\right\} \\ &\leq \rho_3\{|Ka_n - a_{n-1} + a_n - Ka_n| + a_{n-1} - a_0 + |a_n|\rho_3^n\} \\ &\leq \rho_3\{Ka_n + (K-1)|a_n| - a_0 + \rho_3^n|a_n|\}. \end{aligned}$$

Since $h(0) = 0$, $h(z)$ is analytic in $|z| \leq \rho_3$, by Schwarz lemma we have

$$|h(z)| \leq \{Ka_n + (K-1)|a_n| - a_0 + \rho_3^n|a_n|\}|z|$$

if $|z| \le \rho_3$. Now from (28) we see that for $|z| \le \rho_3$,

$$\begin{aligned}|g(z)| &\ge |a_0| - |h(z)|\\ &\ge |a_0| - \{Ka_n + (K-1)|a_n| - a_0 + \rho_3^n|a_n|\}|z|\\ &> 0,\end{aligned}$$

if

$$|z| < \frac{|a_0|}{Ka_n + (K-1)|a_n| - a_0 + \rho_3^n|a_n|}.$$

This proves the theorem completely. □

Proof of Theorem 3. Consider the polynomial

$$\begin{aligned}g(z) &= (1-z)q(z)\\ &= a_0 + \sum_{j=1}^{p-1}(a_j - a_{j-1})z^j + (a_p - a_{p-1})z^p\\ &\quad + \sum_{j=p+1}^{n}(a_j - a_{j-1})z^j - a_n z^{n+1}\\ &= \phi(z) + \psi(z),\end{aligned} \tag{29}$$

where

$$\phi(z) = a_0 + \sum_{j=1}^{p-1}(a_j - a_{j-1})z^j$$

and

$$\psi(z) = (a_p - a_{p-1})z^p + \sum_{j=p+1}^{n}(a_j - a_{j-1})z^j - a_n z^{n+1}.$$

Now for $|z| = \rho_5$ ($\rho_5 < 1$),

$$\begin{aligned}|\psi(z)| &\ge |a_p - a_{p-1}|\rho_5^p - \rho_5^{p+1}\left\{|a_n|\rho_5^{n-p} + \sum_{j=p+1}^{n}|a_j - a_{j-1}|\rho_5^{j-(p+1)}\right\}\\ &\ge |a_p - a_{p-1}|\rho_5^p - \rho_5^{p+1}\left\{|a_n| + \sum_{j=p+1}^{n}|a_j - a_{j-1}|\right\}\\ &= |a_p - a_{p-1}|\rho_5^p - \rho_5^{p+1}\left\{|a_n| + |a_n - a_{n-1}| + \sum_{j=p+1}^{n-1}|a_j - a_{j-1}|\right\}\\ &= |a_p - a_{p-1}|\rho_5^p - \rho_5^{p+1}\left\{|a_n| + |Ka_n - a_{n-1} + a_n - Ka_n|\right.\\ &\qquad\left. + \sum_{j=p+1}^{n-1}|a_j - a_{j-1}|\right\}\end{aligned}$$

$$
\begin{aligned}
&\geq |a_p - a_{p-1}|\rho_5^p - \rho_5^{p+1}\Big\{|a_n| + |Ka_n - a_{n-1}| + (K-1)|a_n| \\
&\qquad\qquad + \sum_{j=p+1}^{n-1} |a_j - a_{j-1}|\Big\} \\
&\geq |a_p - a_{p-1}|\rho_5^p - \rho_5^{p+1}\Big\{K|a_n| + (K|a_n| - |a_{n-1}|)\cos\alpha \\
&\quad + (K|a_n| + |a_{n-1}|)\sin\alpha + \sum_{j=p+1}^{n-1} [(|a_j| - |a_{j-1}|)\cos\alpha \\
&\quad + (|a_j| + |a_{j-1}|)\sin\alpha]\Big\} \quad \text{(by Lemma 1)} \\
&= |a_p - a_{p-1}|\rho_5^p - \rho_5^{p+1}\Big\{K|a_n| + (K|a_n| - |a_p|)\cos\alpha \\
&\quad + (K|a_n| + |a_{n-1}|)\sin\alpha + \sum_{j=p+1}^{n-1} (|a_j| + |a_{j-1}|)\sin\alpha\Big\} \\
&= |a_p - a_{p-1}|\rho_5^p - \rho_5^{p+1} M_4 \\
&= \left(\frac{p}{M_4}\right)^p \left(\frac{|a_p - a_{p-1}|}{p+1}\right)^{p+1}, \quad \text{(by definition of } M_4).
\end{aligned}
$$

Thus on $|z| = \rho_5$,

$$
\begin{aligned}
|\psi(z)| &\geq \left(\frac{p}{M_4}\right)^p \left(\frac{|a_p - a_{p-1}|}{p+1}\right)^{p+1} \\
&> |a_0| + \frac{p}{M_4}\left(\frac{|a_p - a_{p-1}|}{p+1}\right)m' \qquad (30)\\
&= |a_0| + \rho_5 m' \quad \text{(by (18))}.
\end{aligned}
$$

Now for $|z| = \rho_5$,

$$
\begin{aligned}
|\phi(z)| &\leq |a_0| + \sum_{j=1}^{p-1} |a_j - a_{j-1}|\rho_5^j \leq |a_0| + \rho_5 \sum_{j=1}^{p-1} |a_j - a_{j-1}| \\
&\leq |a_0| + \rho_5\Big\{\sum_{j=1}^{p-1}(|a_j| - |a_{j-1}|)\cos\alpha + \sum_{j=1}^{p-1}(|a_j| + |a_{j-1}|)\sin\alpha\Big\} \\
&= |a_0| + \rho_5\Big\{(|a_{p-1}| - |a_0|)\cos\alpha + \sum_{j=1}^{p-1}(|a_j| + |a_{j-1}|)\sin\alpha\Big\} \\
&= |a_0| + \rho_5 m' \quad \text{(by (20))}.
\end{aligned}
$$

Thus for $|z| = \rho_5$,

(31) $$|\phi(z)| \leq |a_0| + \rho_5 m'.$$

From (30) and (31), we see that on $|z| = \rho_5$, $|\phi(z)| < |\psi(z)|$, thereby implying by Rouche's theorem that $g(z) = \phi(z) + \psi(z)$ and $\psi(z)$ have the same number of zeros in $|z| < \rho_5$. Since $\psi(z)$ has at least p zeros in $|z| < \rho_5$, this implies that $g(z)$ and hence $q(z)$ has at least p zeros in $|z| < \rho_5 = \dfrac{p}{p+1}\dfrac{|a_p - a_{p-1}|}{M_4}$.

Thus the proof of Theorem 3 is completed. □

Proof of Theorem 4. It is clear that $\lim_{j\to\infty} a_j = 0$. Consider

(32) $$\begin{aligned} F(z) &= (z-1)f(z) \\ &= -a_0 + (a_0 - a_1)z + (a_1 - a_2)z^2 + \ldots + (a_{p-1} - a_p)z^p + \ldots \\ &= \phi(z) + \psi(z), \end{aligned}$$

where

$$\phi(z) = -a_0 + \sum_{j=1}^{p-1}(a_{j-1} - a_j)z^j$$

and

$$\psi(z) = (a_{p-1} - a_p)z^p + \sum_{j=p+1}^{\infty}(a_{j-1} - a_j)z^j$$

Now for $|z| = \rho_6$ ($\rho_6 < 1$, by hypothesis for $\frac{a_{p-1}}{a_p} < 2 + \frac{1}{p}$),

$$\begin{aligned} |\psi(z)| &\geq |a_{p-1} - a_p|\rho_6^p - \rho_6^{p+1}\left\{\sum_{j=p+1}^{\infty}|a_{j-1} - a_j|\rho_6^{j-(p+1)}\right\} \\ &\geq (a_{p-1} - a_p)\rho_6^p - \rho_6^{p+1}\left\{\sum_{j=p+1}^{\infty}|a_{j-1} - a_j|\right\} \\ &= (a_{p-1} - a_p)\rho_6^p - \rho_6^{p+1}a_p \\ &= \left(\frac{p}{a_p}\right)^p\left(\frac{a_{p-1} - a_p}{p+1}\right)^{p+1} \\ &> |a_0| + \rho_6(a_0 - a_{p-1}) \quad \text{(by (23))}. \end{aligned}$$

Thus for $|z| = \rho_6$,

(33) $$|\psi(z)| > |a_0| + \rho_6(a_0 - a_{p-1}).$$

Now for $|z| = \rho_6$ $(\rho_6 < 1)$

$$\begin{aligned}|\phi(z)| &\leq |a_0| + \sum_{j=1}^{p-1} |a_{j-1} - a_j| \rho_6^j \\ &\leq |a_0| + \rho_6 \sum_{j=1}^{p-1} |a_{j-1} - a_j| \\ &= |a_0| + \rho_6 (a_0 - a_{p-1}).\end{aligned}$$

Now the remaining proof of the Theorem 4 follows on the same lines of Theorem 3.

Acknowledgment. The authors are grateful to the referee for valuable suggestions.

References

[1] Gardner, R. B., Govil, N. K., *Some generalizations of the Eneström–Kakeya theorem*, Acta Math. Hungar. **74** (1–2) (1997), 125–134.

[2] Govil, N. K., Rahman, Q. I., *On the Eneström–Kakeya theorem*, Tôhoku Math. J. **20** (1968), 126–136.

[3] Jain, V. K., *On the zeros of a polynomial*, Proc. Indian Acad. Sci. Math. Sci. **119** (1) (2009), 37—43.

[4] Joyal, A., Labelle, G. and Rahman, Q. I., *On the location of zeros of polynomials*, Canadian Math. Bull. **10** (1967), 55–63.

[5] Marden, M., *The Geometry of Polynomials*, Math. Surveys No. 3, Amer. Math. Soc., Providence, RI, 1966.

[6] Pellet, M.A., *Sur un mode de séparation des racines des équations et la formule de Lagrange*, Bull. Sci. Math. **5** (1881), 393–395.

Roshan Lal
Department of Mathematics
Government Degree College
Chaubattakhal (Pauri)
Uttrakhand 246 162
India
e-mail: rlkeshtwal@yahoo.co.in

Susheel Kumar
Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia (Central University)
New Delhi - 110 025
India
e-mail: ahlawat_skumar@yahoo.co.in

Sunil Hans
Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia (Central University)
New Delhi - 110 025
India
e-mail: sunil.hans82@yahoo.com

Received February 26, 2010