doi: 10.2478/v10062-011-0025-2

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXV, NO. 2, 2011 SECTIO A 203–216

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The Poisson extension of *K*-quasihomography on the unit circle

Dedicated to the memory of Professor Jan Krzyż

Abstract. In this paper some estimates for the Poisson extension of a K-quasihomography on the unit circle are given.

1. Introduction. We say that a real function $\varphi : \mathbb{R} \to \mathbb{R}$ is ρ-quasisymmetric in Beurling–Ahlfors sense [1] for some $\rho \geq 1$ if it satisfies the following condition

$$\frac{1}{\rho} \leq \frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \leq \rho,$$

for every $x, t \in \mathbb{R}$.

We say that a complex function $f : T \to T$, $T \coloneqq \{z : |z| = 1\}$ is ρ-quasisymmetric on the unit circle T if there exists a ρ-quasisymmetric function $\varphi : \mathbb{R} \to \mathbb{R}$, which is a 2π-periodic function and such that

$$f\left(e^{it}\right) = e^{i\varphi(t)}, \qquad t \in \mathbb{R}.$$

The family of all ρ-quasisymmetric functions on the unit circle T will be denoted by $Q_T(\rho)$.

Both of such classes of functions were introduced during the study of the extensions of quasiconformal mappings F from the upper half-plane to

2000 *Mathematics Subject Classification.* 30C62, 30C75, 30C40.

Key words and phrases. Poisson extension, quasiconformal, quasisymmetric, quasihomography, cross-ratio.

its closure, or equivalently from the open unit disc to its closure. A ρ-quasisymmetric function is a restriction of this extension to a border of an appropriate domain.

The ρ-quasisymmetric functions were used for "interpretation" of boundary behaviour of quasiconformal mappings of the half-plane or the unit disc onto itself. The property of such functions were investigated by many authors, for example [1], [7], [6].

We see that such definition of the ρ-quasisymmetric complex mapping on the unit circle is a form of direct transfer of the definition of Ahlfors and Beurling of a ρ-quasisymmetric real mapping on the real axis.

The other "interpretation" of ρ-quasiconformality on the real axis (unit circle, or any Jordan curve) was introduced by J. Zając in [18]. He introduced a point of a K-quasihomography of the unit circle onto itself, if it changes a cross-ratio in a limited way, see Definition 1 below.

1.1. The Poisson extension of an automorphism on the unit circle. By $\mathcal{H}_T$ we will denote the family of all sense preserving automorphisms on the unit circle T. Moreover, by $\mathcal{H}_T^0$ we will denote the subfamily of $\mathcal{H}_T$ consisting of all automorphisms normalized by conditions

$$f(p_k) = p_k, \quad k = 0, 1, 2, \tag{1.1}$$

where $p_k = e^{k \cdot 2\pi i/3}$ are the cube roots of the unity.

It is well known that every $f \in \mathcal{H}_T$ has the extension to the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ given by the Poisson integral $P[f]$

$$P[f](z) := \frac{1}{2\pi} \int_T f(u) \operatorname{Re} \frac{u+z}{u-z} |du|. \tag{1.2}$$

Because the Poisson kernel of the above integral is a real harmonic function, so a function $P[f]$ is a complex harmonic function.

It is well known that for $f \in \mathcal{H}_T$ the extension $P[f]$ is continuous on the closure of D,

$$\lim_{\xi \to z} P[f](\xi) = f(z), \text{ for } z \in T.$$

Therefore, the mapping $P[f]$ can be expressed as a series

$$P[f](z) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \overline{z}^n, \qquad |z| \le 1. \tag{1.3}$$

As shown in [10], [2], for any $f \in \mathcal{H}_T$ and let $F = P[f]$ satisfying (1.3) we have

$$|a_{-n}| \le \frac{n+1}{n\pi} \sin \frac{\pi}{n+1}, \qquad n \in N.$$

Some authors, see for example [12], [8], [9] have considered the Poisson extension $P[f]$ for a ρ-quasisymmetric mapping $f \in Q_T(\rho)$, $\rho \ge 1$.

D. Partyka [12], proved that

$$|a_0| = |P[f](0)| \leq \cos\frac{\pi}{1+\rho}.$$

In another way J. G. Krzyż and M. Nowak [8], obtained the similar result

$$\max\{|a_0|, |a_2|\} \leq \cos\frac{\pi}{1+\rho}.$$

The latter result was obtained by using some special estimates for 2π-periodic functions on the real axis.

In this paper we will give an analogous result when f is K-quasihomography.

In order to estimate the coefficients a_n let us write a function f in power and polar forms

$$f\left(e^{it}\right) = e^{i\alpha(t)} = \cos\alpha(t) + i\sin\alpha(t), \quad t \in \mathbb{R},$$

where a function $\alpha(t)$ is a real 2π-periodic function. Next, let us write the Poisson kernel as a series

$$\begin{aligned}
\operatorname{Re}\frac{u+z}{u-z} &= \frac{1}{2}\left(\frac{u+z}{u-z} + \overline{\left(\frac{u+z}{u-z}\right)}\right) \\
&= \frac{1}{2}\left(1 + 2\sum_{n=1}^{\infty}(\overline{u}z)^n + \overline{\left(1 + 2\sum_{n=1}^{\infty}(\overline{u}z)^n\right)}\right) \\
&= 1 + \sum_{n=1}^{\infty}\overline{u}^n z^n + \sum_{n=1}^{\infty}u^n\bar{z}^n.
\end{aligned}$$

Using the above series, we get

$$\begin{aligned}
P[f](z) &= \frac{1}{2\pi}\int_T f(u)\operatorname{Re}\frac{u+z}{u-z}|du| \\
&= \frac{1}{2\pi}\int_T f(u)\left(1 + \sum_{n=1}^{\infty}\overline{u}^n z^n + \sum_{n=1}^{\infty}u^n\bar{z}^n\right)|du| \\
&= \frac{1}{2\pi}\int_T f(u)|du| + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi}\int_T f(u)\overline{u}^n|du|\right)z^n \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi}\int_T f(u)u^n|du|\right)\bar{z}^n,
\end{aligned}$$

therefore,

$$a_0 = \frac{1}{2\pi} \int_T f(u)\,|du|, \tag{1.4}$$

$$a_n = \frac{1}{2\pi} \int_T f(u)\,\overline{u}^n\,|du|, \qquad n = 1, 2, \ldots, \tag{1.5}$$

$$a_{-n} = \frac{1}{2\pi} \int_T f(u)\,u^n\,|du|, \qquad n = 1, 2, \ldots. \tag{1.6}$$

Substituting $u = e^{it}$ in (1.4), (1.5) and (1.6), we obtain

$$a_0 = \int_0^{2\pi} e^{i\alpha(t)} dt, \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{i\alpha(t)} e^{-int} dt, \qquad n = \pm 1, \pm 2, \ldots.$$

Moreover,

$$\operatorname{Re} a_0 = \frac{1}{\pi} \int_0^{2\pi} \cos\alpha(t)\,dt, \qquad \operatorname{Im} a_0 = \frac{1}{\pi} \int_0^{2\pi} \sin\alpha(t)\,dt,$$

$$\operatorname{Re} a_n = \frac{1}{2\pi} \int_0^{2\pi} \cos(\alpha(t) - nt)\,dt, \qquad n = \pm 1, \pm 2, \ldots,$$

$$\operatorname{Im} a_n = \frac{1}{2\pi} \int_0^{2\pi} \sin(\alpha(t) - nt)\,dt, \qquad n = \pm 1, \pm 2, \ldots.$$

1.2. The family of K-quasihomographies. A concept of K-quasihomography was introduced by J. Zając in [18] and studied in [19], [17], [15], [16], [14]. A mapping on a circle (or more generally, Jordan curves) is called K-quasihomography if it changes the cross-ratio of four points (in the case of Jordan curves the harmonic cross-ratio) in a limited way.

Recall that the cross-ratio of four distinct points is denoted by $[z_1, z_2, z_3, z_4]$ and defined as

$$[z_1, z_2, z_3, z_4] := \left(\frac{z_3 - z_2}{z_3 - z_1} : \frac{z_4 - z_2}{z_4 - z_1} \right)^{1/2}$$

for $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$.

The cross-ratio is a real number if and only if these points lie on a generalized circle. In addition, if these points are ordered on the circle, then the value of the cross-ratio is a positive number.

The cross-ratio is invariant under a homography, this means that if $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homography, then for arbitrary points $z_1, z_2, z_3, z_4 \in \bar{\mathbb{C}}$ we have

$$[z_1, z_2, z_3, z_4] = [h(z_1), h(z_2), h(z_3), h(z_4)].$$

In the papers [4] and [5] the authors proved that this condition is also a sufficient condition for h to be a homography.

In the theory of quasiconformal mappings a very important role plays the Hersch–Pfluger distortion function Φ_K, $K > 0$, for definition see for example [11].

Definition 1. Let $f \in \mathcal{H}(T)$. We say that f is K-quasihomography for $K \geq 1$ if the following inequalities

$$\Phi_{1/K}\left([z_1, z_2, z_3, z_4]\right) \leq \left[f(z_1), f(z_2), f(z_3), f(z_4)\right] \leq \Phi_K\left([z_1, z_2, z_3, z_4]\right)$$

hold for any ordered four points $z_1, z_2, z_3, z_4 \in T$.

The family of K-quasihomographies is denoted by $\mathcal{A}_T(K)$ and

$$\mathcal{A}_T^0(K) = \{f \in \mathcal{A}_T(K) : f(p_k) = p_k,\ k = 0, 1, 2\}.$$

J. Zając in [18, p. 48–50], investigating the family of K-quasihomographies, showed that this family is significantly different from the family of ρ-quasisymmetric mappings, specifically

$$\mathcal{A}_T^0(1) \setminus Q_T(\rho) \neq \emptyset,$$

it means that for every $\rho \geq 1$ there exists a homography which is not a ρ-quasisymmetric mapping.

Despite this, there are dependencies:

Theorem 2 ([18]). *For each $\rho \geq 1$, there exists a constant $K = K(\rho)$, $K \geq 1$ such that*

$$Q_T(\rho) \subset \mathcal{A}_T(K).$$

If $\rho \to 1^+$ we have also $K(\rho) \to 1^+$.

Theorem 3. *For each $K \geq 1$, there exists a constant $\rho = \rho(K)$, $\rho \geq 1$ such that*

$$\mathcal{A}_T^0(K) \subset Q_T^0(\rho).$$

If $K \to 1^+$ we have also $\rho(K) \to 1^+$.

The details can be found in [18, p. 51]. In the same paper, some special functions η_K, with $K > 0$, have been defined, which will be used in the next results. Quote

$$\begin{aligned}
\eta_K(t) &:= \left(d^{-1} \circ \phi_K \circ d\right)(t), \text{ for } t \in \langle 0, 2\pi/3), \\
\eta_K(t) &:= 2\pi/3 + \eta_K(t - 2\pi/3), \text{ for } t \in \langle 2\pi/3, 4\pi/3), \\
\eta_K(t) &:= 4\pi/3 + \eta_K(t - 4\pi/3), \text{ for } t \in \langle 4\pi/3, 2\pi\rangle
\end{aligned}$$

and

$$\begin{aligned}
d(t) &:= \frac{1}{2}\left(1 + \sqrt{3}\tan(t/2 - \pi/6)\right), \text{ for } t \in \langle 0, 2\pi/3), \\
\phi_K(t) &= \left(\Phi_K\left(\sqrt{t}\right)\right)^2, \quad t \in [0, 1], \quad K > 0.
\end{aligned}$$

The following theorem shows how a K-quasihomography can change the value of $\arg f(z)$.

Theorem 4 ([18]). *For $f \in \mathcal{A}_T^0(K)$ and for any $z \in T$ we have estimates*

$$\eta_{1/K}(\arg z) \leq \arg f(z) \leq \eta_K(\arg z). \tag{1.7}$$

The functions η_K have some similar properties as the functions ϕ_K and Φ_K. Namely: because of $\phi_K(1-t)+\phi_{1/K}(t)=1,$ for $t\in[0,1]$ we have

$$\eta_K(t)+\eta_{1/K}(2\pi-t)=2\pi,\ \text{for } t\in[0,2\pi], \tag{1.8}$$

because of $\phi_K\circ\phi_L=\phi_{K\cdot L}$ we have

$$\eta_K\circ\eta_L=\eta_{K\cdot L}\ .$$

It is also known, [13] that for any $K>0$ the function $t\to|\phi_K(t)-t|$ attains a local maximum at the point

$$s_K=\phi_{1/\sqrt{K}}(1/2)$$

and the maximum value is

$$m_K=2\left(\Phi_K\left(1/\sqrt{2}\right)\right)^2-1.$$

Using the above results, the following theorem was obtained.

Theorem 5 ([13]). *For any $K>0$ the function $t\to\eta_K(t)-t$ attains a local maximum at the points*

$$t_K=\eta_{1/\sqrt{K}}(\pi/3),\quad t_K+2\pi/3,\quad t_K+4\pi/3$$

and the maximum value is

$$y_K=2\eta_{\sqrt{K}}(\pi/3)-2\pi/3.$$

We also know that the function $K\to t_K$ for $K>0$ is decreasing and

$$\lim_{K\to0^+}t_K=2\pi/3,\qquad\lim_{K\to1}t_K=\pi/3,\qquad\lim_{K\to\infty}t_K=0.$$

2. Main results.

2.1. The range of variation of the functional $P[f](0)$ for a K-quasihomography. In this section we give some estimations of a coefficient a_0 of the Poisson extension of a function f which is K-quasihomography on the unit circle T.

Theorem 6. *Let $f\in A^0(K)$ and $F(z)=P[f](z)$ be an extension of f given by the Poisson integral* (1.2) *with the form* (1.3). *Then*

$$-\frac{2}{3}\le B(K)\le\operatorname{Re}a_0\le C(K)\le\frac{1}{2}, \tag{2.1}$$

where

$$C(K)=\begin{cases}\frac{1}{2\pi}\left(\frac{2}{3}\pi-2t_K-\sqrt{3}+2\sqrt{3}\sin(\pi/6+2t_K)\right), & \text{for } 1\le K\le K_0,\\ \frac{1}{2\pi}\left(\pi-4t_K+2\sin2t_K\right), & \text{for } K\ge K_0,\end{cases}$$

$$B(K)=\begin{cases}\frac{1}{2\pi}\left(6t_K+\sqrt{3}-2\pi-2\sin2t_K\right), & \text{for } 1\le K\le K_0,\\ \frac{1}{2\pi}\left(2t_K+\sqrt{3}-\frac{4\pi}{3}-2\sin(\pi/3+2t_K)\right), & \text{for } K\ge K_0,\end{cases}$$

and K_0 *is such that* $t_{K_0} = \pi/6$. *We also have*

$$\lim_{K\to 1^+} C(K) = B(K) = 0$$

and

$$\lim_{K\to\infty} C(K) = \frac{1}{2}, \qquad \lim_{K\to\infty} B(K) = -\frac{2}{3}.$$

Proof of Theorem 6. Recall that $\operatorname{Re} a_0 = \frac{1}{\pi}\int_0^{2\pi} \cos\alpha(t)\,dt$, where $\alpha(t) = \arg f\left(e^{it}\right)$. In order to estimate the coefficient a_0 we use the estimate of a function $\alpha(t)$ given in (1.7) and an estimate of the function η_K with some properly chosen linear functions.

Due to the estimate (1.7) we have

$$\cos\alpha(t) \le \max\left\{\cos\eta_K(t), \cos\eta_{1/K}(t)\right\} = \begin{cases} \cos\eta_{1/K}(t), & \text{for } t\in[0,\pi] \\ \cos\eta_K(t), & \text{for } t\in(\pi,2\pi], \end{cases}$$

see Fig. 1.

By making use of above estimate, properties of the function η_K given in (1.8) and integration by a substitution, we obtain

$$\begin{aligned} \operatorname{Re} a_0 &\le \frac{1}{2\pi}\left(\int_0^{\pi} \cos\eta_{1/K}(t)\,dt + \int_{\pi}^{2\pi} \cos\eta_K(t)\,dt\right) \\ &= \frac{1}{\pi}\int_0^{\pi} \cos\eta_{1/K}(t)\,dt. \end{aligned} \tag{2.2}$$

Next, let us estimate the function $\eta_{1/K}$ by a partially linear function. From Theorem 5 we can calculate that the linear function

$$y_2(t) = t - 2\pi/3 + 2t_K$$

is a lower bound for η_K for $K \ge 1$. We know also that $\eta_K(2k\pi/3) = \eta_{1/K}(2k\pi/3) = 2k\pi/3$, $k = 0,1,2,3$, therefore

$$\pi \ge \eta_{1/K}(t) \ge t - 2\pi/3 + 2t_K, \quad \text{for } t\in[0,\pi]$$

and

$$\begin{aligned} \eta_{1/K} &\ge 0, \quad \text{for } t\in[0,2t_K), \\ \eta_{1/K}(t) &\ge 2\pi/3, \quad \text{for } t\in(2\pi/3, 2\pi/3+2t_K]. \end{aligned}$$

By the monotonicity of the function $y = \cos t$ in some intervals and above inequalities we have

$$\cos\alpha(t) \le \begin{cases} 1, & \text{for } t\in(0, 2\pi/3 - 2t_K), \\ \cos(t - 2\pi/3 + 2t_K), & \text{for } t\in(2\pi/3 - 2t_K, 2\pi/3), \\ -1/2, & \text{for } t\in(2\pi/3, 4\pi/3 - 2t_K) \\ \cos(t - 2\pi/3 + 2t_K), & \text{for } t\in(4\pi/3 - 2t_K, \pi), \end{cases} \tag{2.3}$$

for K satisfying

$$4\pi/3 - 2t_K \le \pi.$$

This means that the inequalities (2.3) hold for $1 \le K \le K_0$, where

$$t_{K_0} = \pi/6.$$

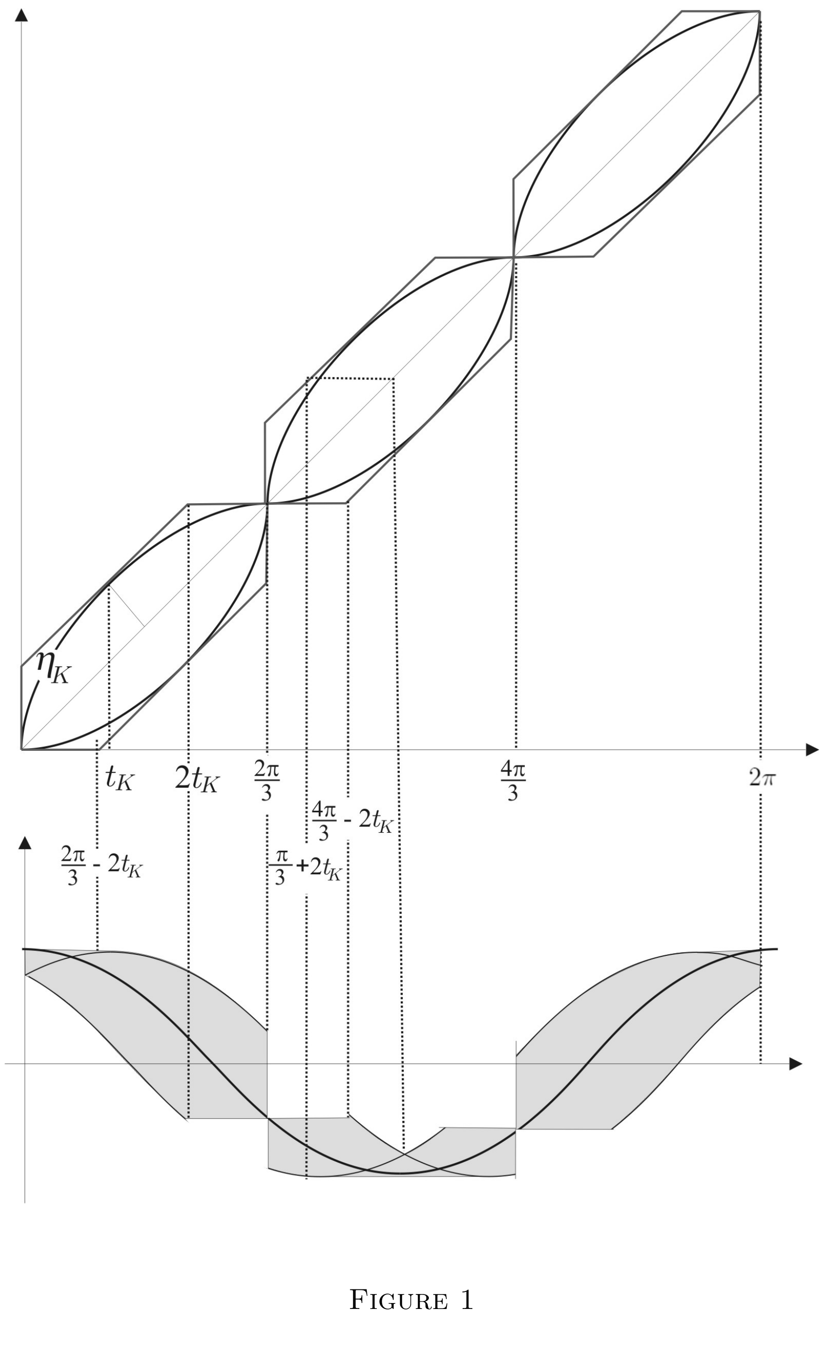

Figure 1

However, for $K > K_0$ we have

$$\cos\alpha\left(t\right) \le \begin{cases} 1, & \text{for } t \in \left(0, 2\pi/3 - 2t_K\right), \\ \cos\left(t - 2\pi/3 + 2t_K\right), & \text{for } t \in \left(2\pi/3 - 2t_K, 2\pi/3\right), \\ -1/2, & \text{for } t \in \left(2\pi/3, \pi\right). \end{cases}$$

We can proceed to estimate the integral (2.2) in two cases, for $1 \le K \le K_0$ and next for $K > K_0$.

For $1 \le K \le K_0$

$$\begin{aligned} \operatorname{Re} a_0 \le& \frac{1}{\pi}\left(\int_0^{2\pi/3-2t_K} 1dt + \int_{2\pi/3-2t_K}^{2\pi/3} \cos\left(t - 2\pi/3 + 2t_K\right)dt\right) \\ &+ \frac{1}{\pi}\left(\int_{2\pi/3}^{4\pi/3-2t_K} \left(-1/2\right)dt + \int_{4\pi/3-2t_K}^{\pi} \cos\left(t - 2\pi/3 + 2t_K\right)dt\right) \\ =& \frac{1}{\pi}\left(\frac{2}{3}\pi - 2t_K + \frac{1}{2}\sqrt{3}\sin\left(\frac{1}{6}\pi + 2t_K\right) - \frac{1}{2}\cos\left(\frac{1}{6}\pi + 2t_K\right)\right) \\ &+ \frac{1}{\pi}\left(t_K - \frac{1}{3}\pi + \frac{1}{2}\sin 2t_K - \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{3}\cos 2t_K\right) \\ =& \frac{1}{\pi}\left(\frac{1}{3}\pi - t_K + \frac{3}{2}\sin 2t_K - \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{3}\cos 2t_K\right) \\ =& \frac{1}{\pi}\left(\frac{1}{3}\pi - t_K - \frac{1}{2}\sqrt{3} + \sqrt{3}\left(\frac{\sqrt{3}}{2}\sin 2t_K + \frac{1}{2}\cos 2t_K\right)\right) \\ =& \frac{1}{2\pi}\left(2\pi/3 - 2t_K - \sqrt{3} + 2\sqrt{3}\sin\left(\pi/6 + 2t_K\right)\right) =: C_1\left(K\right). \end{aligned}$$

However, for $K > K_0$

$$\begin{aligned} \operatorname{Re} a_0 \le& \frac{1}{\pi}\left(\int_0^{2\pi/3-2t_K} 1dt + \int_{2\pi/3-2t_K}^{2\pi/3} \cos\left(t - 2\pi/3 + 2t_K\right)dt\right. \\ &\left. + \int_{2\pi/3}^{\pi}\left(-\frac{1}{2}\right)dt\right) \\ =& \frac{1}{2\pi}\left(\pi - 4t_K + 2\sin 2t_K\right) =: C_2\left(K\right). \end{aligned}$$

Note that

$$\lim_{K\to 1^+} C_1(K) = \lim_{t_K\to\pi/3} \frac{1}{2\pi}\left(2\pi/3 - 2t_K - \sqrt{3} + 2\sqrt{3}\sin(\pi/6 + 2t_K)\right) = 0,$$

$$\lim_{K\to K_0^-} C_1(K) = \lim_{t_K\to\pi/6^+} \frac{1}{2\pi}\left(2\pi/3 - 2t_K - \sqrt{3} + 2\sqrt{3}\sin(\pi/6 + 2t_K)\right)$$
$$= \frac{\sqrt{3}}{2\pi} + \frac{1}{6},$$

$$\lim_{K\to K_0^+} C_2(K) = \lim_{t_K\to\pi/6^-} \frac{1}{2\pi}(\pi - 4t_K + 2\sin 2t_K) = \frac{\sqrt{3}}{2\pi} + \frac{1}{6},$$

$$\lim_{K\to\infty} C_2(K) = \lim_{t_K\to 0^+} \frac{1}{2\pi}(\pi - 4t_K + 2\sin 2t_K) = \frac{1}{2}.$$

We obtain

$$C(K) = \begin{cases} C_1(K), & \text{for } 1 \le K \le K_0, \\ C_2(K), & \text{for } K \ge K_0 \end{cases}$$

and

$$\operatorname{Re} a_0 \le C(K).$$

Similarly,

$$\cos\alpha(t) \ge \min\left\{\cos\eta_K(t), \cos\eta_{1/K}(t)\right\},$$
$$\ge \begin{cases} \cos\eta_K(t), & \text{for } t \in \left[0, \eta_{1/K}(\pi)\right], \\ -1, & \text{for } t \in \left(\eta_{1/K}(\pi), \eta_K(\pi)\right) \\ \cos\eta_K(t), & \text{for } t \in [\eta_K(\pi), 2\pi]. \end{cases}$$

The straight line

$$y_1(t) = t + 2\pi/3 - 2t_K$$

is an upper bound of the graph of the function η_K. We have

$$\eta_K(t) \le t + 2\pi/3 - 2t_K, \quad \text{for } t \in [0, \pi]$$

and

$$\eta_K \le 2\pi/3, \quad \text{for } t \in (2t_K, 2\pi/3).$$

By the monotonicity of the function $y = \cos t$ and above inequalities we get for $1 \le K \le K_0$ that

$$\cos\alpha(t) \ge \begin{cases} \cos(t + 2\pi/3 - 2t_K), & \text{for } t \in (0, 2t_K), \\ -1/2, & \text{for } t \in (2t_K, 2\pi/3), \\ \cos(t + 2\pi/3 - 2t_K), & \text{for } t \in (2\pi/3, 2t_K + \pi/3), \\ -1, & \text{for } t \in (2t_K + \pi/3, \pi). \end{cases}$$

while for $K > K_0$ we have

$$\cos\alpha(t) \geq \begin{cases} \cos(t + 2\pi/3 - 2t_K), & \text{for } t \in (0, 2t_K), \\ -1/2, & \text{for } t \in (2t_K, 2\pi/3), \\ -1, & \text{for } t \in (\pi/3, \pi). \end{cases}$$

Consequently, for $1 \leq K \leq K_0$

$$\begin{aligned} \operatorname{Re} a_0 &\geq \frac{1}{\pi}\left(\int_0^{2t_K} \cos(t + 2\pi/3 - 2t_K)\,dt + \int_{2t_K}^{2\pi/3} (-1/2)\,dt\right) \\ &\quad + \frac{1}{\pi}\left(\int_{2\pi/3}^{\pi/3+2t_K} \cos(t + 2\pi/3 - 2t_K)\,dt + \int_{\pi/3+2t_K}^{\pi} (-1)\,dt\right) \\ &= \frac{1}{2\pi}\left(6t_K - 2\pi + \sqrt{3} - 2\sin 2t_K\right) =: S_1(K), \end{aligned}$$

and for $K \geq K_0$

$$\begin{aligned} \operatorname{Re} a_0 \geq \frac{1}{\pi}\Bigg(&\int_0^{2t_K} \cos(t + 2\pi/3 - 2t_K)\,dt \\ &+ \int_{2t_K}^{2\pi/3} (-1/2)\,dt + \int_{2\pi/3}^{\pi} (-1)\,dt\Bigg) \\ = \frac{1}{2\pi}&\left(2t_K + \sqrt{3} - \frac{4\pi}{3} - 2\sin(\pi/3 + 2t_K)\right) =: S_2(K). \end{aligned}$$

Note that

$$\lim_{K\to 1^+} B_1(K) = \lim_{t_K\to\pi/3} \frac{1}{2\pi}\left(6t_K - 2\pi + \sqrt{3} - 2\sin 2t_K\right) = 0,$$

$$\lim_{K\to K_0^-} B_1(K) = \lim_{t_K\to\pi/6^+} \frac{1}{2\pi}\left(6t_K - 2\pi + \sqrt{3} - 2\sin 2t_K\right) = -\frac{1}{2},$$

$$\lim_{K\to K_0^+} B_2(K) = \lim_{t_K\to\pi/6^-} \frac{1}{2\pi}\left(2t_K + \sqrt{3} - \frac{4\pi}{3} - 2\sin(\pi/3 + 2t_K)\right) = -\frac{1}{2},$$

$$\lim_{K\to\infty} B_2(K) = \lim_{t_K\to 0^+} \frac{1}{2\pi}\left(2t_K + \sqrt{3} - \frac{4\pi}{3} - 2\sin(\pi/3 + 2t_K)\right) = -\frac{2}{3}.$$

Finally

$$\operatorname{Re} a_0 \geq B(K),$$

which completes the proof of the inequality (2.1). □

Theorem 7. *Let $f \in A^0(K)$ and $F(z) = P[f](z)$ be an extension of f given by the Poisson integral* (1.2) *with the form* (1.3). *Then*

$$|a_0| \leq \frac{2}{\sqrt{3}}\sqrt{(C(K))^2 + C(K)B(K) + (B(K))^2} =: R(K)$$

and also

$$\lim_{K\to 1^+} R(K) = 0,$$
$$\lim_{K\to\infty} R(K) = \sqrt{\frac{39}{81}} < 0.694.$$

Proof of Theorem 7. We note first that the inequalities (2.1) mean that the expression $a_0 = P[f](0)$ lies in a strip

$$S(K) \coloneqq \{x + iy : B(K) \le x \le C(K)\}.$$

Moreover, due to the normalization conditions (1.1) it can be concluded that the expression a_0 lies also in the strip $e^{2\pi i/3}S(K)$ and in the strip $e^{4\pi i/3}S(K)$, which arise from the strip $S(K)$ by the rotation by the angles $2\pi/3$ and $4\pi/3$. In general, such common part of these three strips may be a triangle or a hexagon. Let us put

$$H(K) \coloneqq S(K) \cap e^{2\pi i/3}S(K) \cap e^{4\pi i/3}S(K).$$

The condition for $H(K)$ to be a hexagon is

$$\frac{1}{2} \le \left|\frac{B(K)}{C(K)}\right| \le 2.$$

It is easy to check that this condition is done in both cases (for $1 \le K \le K_0$ and $K > K_0$). Therefore, the expression a_0 is located in the hexagon $H(K)$. From elementary geometrical dependence it can be noted that the hexagon $H(K)$ has six symmetry axes and is inscribed in the circle of radius $R(K)$ equal to

$$R(K) = \frac{2}{\sqrt{3}}\sqrt{C(K)^2 + C(K)B(K) + B(K)^2}.$$

Note that

$$\lim_{K\to 1^+} R(K) = 0,$$
$$\lim_{K\to K_0} R(K) = \frac{1}{2\pi}\sqrt{\frac{28}{27}\pi^2 - \frac{4}{9}\pi\sqrt{3} + 4} < 0.547,$$
$$\lim_{K\to\infty} R(K) = \frac{1}{9}\sqrt{39} < 0.694. \qquad \square$$

2.2. Some remarks on extensions of the normalized automorphisms of the unit circle. Suppose that $f \in \mathcal{H}^0(K)$ is an automorphism on the unit circle T, normalized by conditions (1.1). Taking into account the normalization (1.1) and using (1.3), we obtain

$$1 = \sum_{n=-\infty}^{\infty} a_n,$$

$$e^{2\pi i/3} = \sum_{n=-\infty}^{\infty} a_n e^{2\pi n i/3},$$

$$e^{4\pi i/3} = \sum_{n=-\infty}^{\infty} a_n e^{4\pi n i/3}.$$

Denote $A_k = \sum_{n=1}^{\infty} a_{3n+k}$, $k = 0, 1, 2$, from this we get

$$\begin{cases} 1 = A_0 + A_1 + A_2 \\ e^{2\pi i/3} = A_0 + e^{2\pi i/3} A_1 + e^{-2\pi i/3} A_2 \\ e^{-2\pi i/3} = A_0 + e^{-2\pi i/3} A_1 + e^{2\pi i/3} A_2. \end{cases}$$

Solving the above system of equations, we get

$$A_0 = 0, \qquad A_1 = 1, \qquad A_2 = 0. \tag{2.4}$$

The equalities (2.4) we can adopt to the following form

$$\sum_{n=-\infty}^{\infty} a_{3n} = 0, \qquad \sum_{n=-\infty}^{\infty} a_{3n+1} = 1, \qquad \sum_{n=-\infty}^{\infty} a_{3n+2} = 0.$$

Of course,

$$\sum_{n=-\infty}^{\infty} a_n = 1.$$

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Received March 30, 2011