

RAMI AHMAD EL-NABULSI

## On Perelman’s functional with curvature corrections

ABSTRACT. In recent ten years, there has been much concentration and increased research activities on Hamilton’s Ricci flow evolving on a Riemannian metric and Perelman’s functional. In this paper, we extend Perelman’s functional approach to include logarithmic curvature corrections induced by quantum effects. Many interesting consequences are revealed.

During the last decades, there has been more attention focused on the Ricci flow which was introduced in 1982 by Hamilton [19, 20, 21] and extended later by Perelman [28, 26, 27]. In fact, the Ricci flow is a system of the 2nd order nonlinear weakly parabolic partial differential equations (PDEs) on the metric which can be viewed as a nonlinear heat equation of a metric and contains at most the second derivative of the metric which led Perelman to the proof of the famous Thurston’s geometrization conjecture [29]. In fact, Ricci flow plays a crucial role in string theory [5] as it describes the flow energy effective action. Besides, the Ricci flow plays a significant role in the thermodynamics of black holes [22] which is one of the most important objects in string theory. Strings, higher order curvature and black holes have been found to be inextricably knotted [25]. In string theory, there are higher curvature corrections in addition to the Einstein–Hilbert term. Different forms of curvature corrections may be added [6], nevertheless, a logarithmic correction of the form may be induced by quantum effects and

this is quite interesting. It is thus natural to extend the Perelman's functionals to include logarithmic correction of the Ricci scalar and discuss their main properties. It is noteworthy that one of the most motivating questions in quantum gravity concerns the topological change due to quantum effects, a consequence recognized as quantum foam [7, 8, 9, 16, 1]. Topological change in general is an important topic in differential topology with surgery and mainly Ricci flow. This will be the main topic discussed in this paper. We start by introducing the following definition.

**Definition 1.** Let  $\mathcal{M}$  denote an infinite-dimensional manifold with smooth Riemannian metrics  $g$  and smooth metric tensor  $g_{ij}$  evolving in a Riemannian closed manifold  $M$ . The tangent space  $T_g M$  consists of the symmetric covariant 2-tensors  $\nu_{ij} \in T_g M$  on  $M$ . The extended functional  $\mathfrak{F} : \mathcal{M} \times C^\infty(M) \rightarrow \mathbb{R}$  is given by:

$$(1) \quad \mathfrak{F}[f, g_{ij}] := \int_M e^{-f} \left[ |\nabla f|^2 + R + a \ln R \right] dV,$$

where  $f : M \rightarrow \mathbb{R}$  and  $a$  should be chosen so as to have proper dimensions.

**Theorem 1.** *We have*

$$(2) \quad \begin{aligned} \delta \mathfrak{F}[f, g] = \int_M e^{-f} \left[ -\nu_{ij} \left( \left(1 - \frac{a}{R}\right) R_{ij} + \left(1 - \frac{a}{R}\right) \nabla_i \nabla_j f + \frac{a}{R} \nabla_i f \nabla_j f \right) \right. \\ \left. + \left( \frac{\nu}{2} - h \right) \left( 2\Delta f - |\nabla f|^2 - \frac{2a}{R} (\Delta f - |\nabla f|^2) + R \right) \right] dV. \end{aligned}$$

**Proof.** We follow the arguments of [24]. Let  $h = \delta f$ ,  $\nu = g^{ij} \nu_{ij}$ ,  $\delta(dV) = \nu dV/2$  where  $dV = \sqrt{\det g} \prod_{i=1}^n dx_i$ , then the following equalities hold:

$$(3) \quad \delta R = -\Delta \nu + \nabla_i \nabla_j \nu_{ij} - R_{ij} \nu_{ij},$$

$$(4) \quad \delta \left( e^{-f} \right) dV = \left( \frac{\nu}{2} - h \right) e^{-f} dV,$$

$$(5) \quad \delta |\nabla f|^2 = -\nu^{ij} \nabla_i f \nabla_j f + 2 \langle \nabla f, \nabla h \rangle,$$

$$(6) \quad \delta(\ln R) = -\frac{1}{R} \delta R = -\frac{1}{R} (-\Delta \nu + \nabla_i \nabla_j \nu_{ij} - R_{ij} \nu_{ij}),$$

$$(7) \quad \Delta e^{-f} = (|\nabla f|^2 - \Delta f) e^{-f},$$

$$(8) \quad \int_M e^{-f} (-\Delta \nu) dV = \int_M e^{-f} (\Delta f - |\nabla f|^2) \nu dV,$$

$$(9) \quad \int_M e^{-f} \nabla_i \nabla_j \nu_{ij} dV = \int_M e^{-f} (\nabla_i f \nabla_j f - \nabla_i \nabla_j f) \nu_{ij} dV,$$

$$(10) \quad 2 \int_M e^{-f} \langle \nabla f, \nabla h \rangle dV = 2 \int_M e^{-f} (|\nabla f|^2 - \Delta f) h dV.$$

Making use of equalities (3)–(10), we obtain the required result.  $\square$

**Remark 1.** When  $a = 0$ , equality (2) is reduced to the Perelman's main result [28].

By fixing a measure  $dm = e^{-f} dV$  which should be constant in time and taking  $\nu = 2h$ , we obtain

$$(11) \quad \delta \mathfrak{F}[\nu] = \int_M \left[ -\nu_{ij} \left( \left(1 - \frac{a}{R}\right) R_{ij} + \left(1 - \frac{a}{R}\right) \nabla_i \nabla_j f + \frac{a}{R} \nabla_i f \nabla_j f \right) \right] dm,$$

and the gradient flow is

$$(12) \quad \frac{\partial g_{ij}}{\partial t} = -2 \left( \left(1 - \frac{a}{R}\right) R_{ij} + \left(1 - \frac{a}{R}\right) D^2 f + \frac{a}{R} (Df)^2 \right),$$

where  $D \triangleq \nabla_i$ .

**Corollary 1.** *f evolves according to the following modified heat equation:*

$$(13) \quad \frac{\partial f}{\partial t} = - \left(1 - \frac{a}{R}\right) R - \left(1 - \frac{a}{R}\right) \Delta f - \frac{a}{R} (\nabla f)^2.$$

**Proof.** Making use of the fact that  $2\partial f/\partial t = \text{tr}(\partial g/\partial t)$ , we obtain straightforwardly the required result.  $\square$

**Corollary 2.** *If f and g evolve according to equations (12) and (13), then*

$$(14) \quad \frac{d\mathfrak{F}}{dt} = 2 \int_M \left| \left(1 - \frac{a}{R}\right) R_{ij} + \left(1 - \frac{a}{R}\right) D^2 f + \frac{a}{R} (Df)^2 \right|^2 e^{-f} dV.$$

The proof is direct.

**Corollary 3.** *The flow (14) is equivalent to the Ricci flow with logarithmic curvature correction.*

**Proof.** Let  $\phi_t$  be the flow generated by  $\nabla f$  and let  $\bar{g} = \phi_t g$  and  $\bar{f} = f \circ \phi_t$ . Then

$$(15) \quad \frac{d\bar{g}}{dt} = \phi_t^* \left( \frac{\partial g}{\partial t} + \mathcal{L}_{\nabla f} g \right)$$

$$(16) \quad = \phi_t^* \left( -2 \left( \left(1 - \frac{a}{R}\right) R_{ij} + \left(1 - \frac{a}{R}\right) D^2 f + \frac{a}{R} (Df)^2 \right) + 2D^2 f \right)$$

$$(17) \quad = \phi_t^* \left( -2 \left( \left(1 - \frac{a}{R}\right) R_{ij} - \frac{a}{R} D^2 f + \frac{a}{R} (Df)^2 \right) \right)$$

$$(18) \quad = -2 \left( \left(1 - \frac{a}{R}\right) \bar{R}_{ij} - \frac{a}{R} \bar{D}^2 f + \frac{a}{R} (\bar{D}f)^2 \right),$$

where

$$(19) \quad (\mathcal{L}_X g)_{ij} = g \left( \nabla_{\frac{\partial}{\partial x_i}} X, \frac{\partial}{\partial x_j} \right) + g \left( \nabla_{\frac{\partial}{\partial x_j}} X, \frac{\partial}{\partial x_i} \right)$$

is the Lie derivative. Besides,

$$(20) \quad \frac{\partial \bar{f}}{\partial t} = \frac{\partial f}{\partial t} \circ \phi_t + \langle \widehat{\nabla} f, \dot{\phi}_t \rangle \circ \phi_t$$

$$(21) \quad = - \left\{ \left(1 - \frac{a}{R}\right) R + \left(1 - \frac{a}{R}\right) \Delta f + \frac{a}{R} (\nabla f)^2 \right\} \circ \phi_t + \langle \widehat{\nabla} f, \dot{\phi}_t \rangle \circ \phi_t$$

$$(22) \quad = -R \circ \phi_t + a \circ \phi_t - \Delta f \circ \phi_t + \frac{a}{R} \Delta f \circ \phi_t - \frac{a}{R} (\nabla f)^2 \circ \phi_t + \langle \widehat{\nabla} f, \dot{\phi}_t \rangle \circ \phi_t$$

$$(23) \quad = - \left(1 - \frac{a}{R}\right) \bar{R} - \left(1 - \frac{a}{R}\right) \bar{\Delta} f - \frac{a}{R} (\bar{\nabla} f)^2 + |d\bar{f}|^2_{\bar{g}},$$

where  $\bar{R}$  and  $\bar{\Delta} f$  are respectively the curvature and Laplacian of  $\bar{g}$ .  $\square$

**Theorem 2.** *If  $f$  and  $g$  evolve respectively according to the flow:*

$$(24) \quad \frac{\partial f}{\partial t} = - \left(1 - \frac{a}{R}\right) R - \left(1 - \frac{a}{R}\right) \Delta f + \left(1 - \frac{a}{R}\right) |\nabla f|^2$$

$$(25) \quad = -R - \Delta f + |\nabla f|^2 + \frac{a}{R} (R + \Delta f - |\nabla f|^2),$$

$$(26) \quad \frac{\partial g_{ij}}{\partial t} = -2 \left( \left(1 - \frac{a}{R}\right) R_{ij} - \frac{a}{R} \Delta f + \frac{a}{R} (\nabla f)^2 \right)$$

$$(27) \quad = -2R_{ij} + \frac{2a}{R} (R_{ij} + \Delta f - (\nabla f)^2),$$

then

$$(28) \quad \frac{d\mathfrak{F}}{dt} = 2 \int_M \left| \left(1 - \frac{a}{R}\right) R_{ij} + \left(1 - \frac{a}{R}\right) D^2 f + \frac{a}{R} (Df)^2 \right|^2 e^{-f} dV,$$

and in particular  $\mathfrak{F}[f(t), g(t)]$  is non-decreasing and the monotonicity is strict unless

$$(29) \quad R_{ij} + D^2 f - \frac{a}{R} (R_{ij} + D^2 f - (Df)^2) = 0.$$

**Remark 2.** Under the flow represented by equations (23) and (24),  $\int_M e^{-f} dV$  remains constant with time.

A controlled quantity for the Ricci flow is obtained if we define  $\lambda(\mathbf{M}, g) = \inf_M (M, g, f) / f : \int_M e^{-f} dV = 1$ . Hence

$$(30) \quad \lambda(g_{ij}(t)) \leq \mathfrak{F}[f(t), g_{ij}(t)] \leq \mathfrak{F}[f(t_0), g_{ij}(t_0)] = \lambda(g_{ij}(t_0)),$$

for  $t < t_0$ .

**Theorem 3.** *Given a steady breather solution of the Ricci flow on an interval  $[t_1, t_2]$  satisfying the equation  $g(t_2) = \phi^*g(t_1)$  for some  $\phi \in \text{Diff}(\mathbf{M})$ . Then for  $(Df)^2 = 0$  and  $R_{ij} + D^2f = 0$ , a steady breather is a gradient steady soliton and is independent of quantum effects.*

**Proof.** We have  $\lambda(g(t_2)) = \lambda(g(t_1))$ , then  $\mathfrak{F}[f(t), g(t)]$  must be constant in time. From equations (27) and (28), we conclude for  $R \neq 0$ :

$$(31) \quad \frac{\partial f}{\partial t} = |\nabla f|^2,$$

$$(32) \quad \frac{\partial g_{ij}}{\partial t} = 2D^2f = -2R_{ij}.$$

These solutions are independent of  $a$  and hence of quantum effects. The proof is then complete.  $\square$

**Remark 3.** More generally, we can write using equation (28):

$$(33) \quad \frac{\partial f}{\partial t} = |\nabla f|^2,$$

$$(34) \quad \frac{\partial g_{ij}}{\partial t} = 2\Delta f = -2R_{ij} - \frac{2a}{R-a}(\nabla f)^2$$

$$(35) \quad \underset{R \gg a}{\approx} -2R_{ij} - \frac{2a}{R}(\nabla f)^2,$$

$$(36) \quad \underset{R \ll a}{\approx} -2R_{ij} + 2(\nabla f)^2.$$

We can write equation (28) like

$$(37) \quad \nabla_i \nabla_j f + \frac{a}{R-a} \nabla_i f \nabla_j f = -R_{ij}.$$

Taking the covariant derivatives and using  $\nabla_i(R-a)^{-1} = -(R-a)^{-2} \nabla_i R = 2(R-a)^{-2} \nabla_j R_{ij}$ , we get

$$(38) \quad \nabla_i \nabla_j \nabla_k f + \frac{a}{R-a} \nabla_i \nabla_j f \nabla_k f + \frac{2a}{(R-a)^2} \nabla_j R_{ij} \nabla_j f \nabla_k f = -\nabla_i R_{jk},$$

$$(39) \quad \nabla_j \nabla_i \nabla_k f + \frac{a}{R-a} \nabla_j \nabla_i f \nabla_k f + \frac{2a}{(R-a)^2} \nabla_i R_{ji} \nabla_i f \nabla_k f = -\nabla_j R_{ik}.$$

Subtracting equation (38) from (37), we obtain

$$(40) \quad \begin{aligned} & \nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f + \frac{a}{R-a} (\nabla_i \nabla_j f \nabla_k f - \nabla_j \nabla_i f \nabla_k f) \\ & + \frac{2a}{(R-a)^2} (\nabla_j R_{ij} \nabla_j f \nabla_k f - \nabla_i R_{ji} \nabla_i f \nabla_k f) = \nabla_j R_{ik} - \nabla_i R_{jk}. \end{aligned}$$

Making use of the commutating relation

$$(41) \quad \nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = R_{ijkl} \nabla_l f,$$

we find

$$(42) \quad \begin{aligned} \nabla_j R_{ik} - \nabla_i R_{jk} &= R_{ijkl} \nabla_l f + \frac{a}{R-a} (\nabla_i \nabla_j f - \nabla_j \nabla_i f) \nabla_k f \\ &+ \frac{2a}{(R-a)^2} (\nabla_j R_{ij} \nabla_j f - \nabla_i R_{ji} \nabla_i f) \nabla_k f. \end{aligned}$$

After taking the trace on  $j$  and  $k$ , and making use  $\nabla_i R = 2\nabla_j R_{ij}$ , we find

$$(43) \quad \begin{aligned} \nabla_i R &= 2R_{ij} \nabla_j f - \frac{2a}{R-a} (\nabla_i \nabla_j f - \nabla_j \nabla_i f) \nabla_j f \\ &- \frac{4a}{(R-a)^2} \left( \frac{1}{2} \nabla_i R \nabla_j f - \frac{1}{2} \nabla_j R \nabla_i f \right) \nabla_j f. \end{aligned}$$

Accordingly,

$$(44) \quad \begin{aligned} \nabla_i (|\nabla f|^2 + R) &= 2\nabla_j f \left( \nabla_i \nabla_j f + R_{ij} - \frac{a}{R-a} (\nabla_i \nabla_j f - \nabla_j \nabla_i f) \right. \\ &\quad \left. - \frac{a}{(R-a)^2} (\nabla_i R \nabla_j f - \nabla_j R \nabla_i f) \right). \end{aligned}$$

For  $|\nabla f|^2 + R = 0$ , we need either  $\nabla_j f = 0$  or

$$(45) \quad \begin{aligned} \nabla_i \nabla_j f + R_{ij} - \frac{a}{R-a} (\nabla_i \nabla_j f - \nabla_j \nabla_i f) \\ - \frac{a}{(R-a)^2} (\nabla_i R \nabla_j f - \nabla_j R \nabla_i f) = 0. \end{aligned}$$

Taking now the difference of equation  $|\nabla f|^2 + R = 0$  and the trace of equation (36), we find effortlessly:

$$(46) \quad \Delta f - |\nabla f|^2 = \frac{a}{a-R} (\nabla f)^2.$$

If the condition (44) is fulfilled, taking its trace, we get  $\Delta f + R = 0$  and then

$$(47) \quad R = -\Delta f = -\frac{a}{a-R} (\nabla f)^2 - |\nabla f|^2.$$

Then it follows that

$$(48) \quad \begin{aligned} - \int_M \Delta(e^{-f}) dV &= \int_M (\Delta f - |\nabla f|^2) e^{-f} dV \\ &= a \int_M \frac{1}{a-R} (\nabla f)^2 e^{-f} dV \neq 0. \end{aligned}$$

After integrating equation (45), it follows that  $f$  is not a constant and  $g_{ij}$  is not a Ricci flat. In fact, there exist non-compact steady gradient Ricci solitons that are not Ricci flat like the 2-dimensional Hamilton cigar soliton [19, 20, 21, 3, 2].

**Proposition 1.** *Let  $g_{ij}$  be a complete steady gradient Ricci soliton on a manifold  $M$  so that equality (36) holds. Then a gradient steady or expanding Ricci soliton is not necessarily an Einstein metric unless quantum effects due to the logarithmic curvature corrections are absent.*

The problem of the topological change at the quantum level is a serious affair and is connected with the Ricci flows. This issue may be addressed making use of different schemes, e.g. Euclidean path integral approach [17]. In this work, we investigated the topic by adding a logarithmic curvature correction to Perelman's  $\mathfrak{F}$  functional. Quantum corrections can be treated in a variety of ways; however, we have shown that a logarithmic one may lead to many interesting consequences. The topic of Ricci flow with quantum corrections is open and consequently much work is needed. Nevertheless, let us briefly recall that quantum corrections play a leading role in noncommutative geometry. To this end, noncommutative generalization of the Ricci flow was discussed recently by Vacaru in [30]. Vacaru's work is highly important and motivated as it was revealed that unification may exist between the spectral triple action approach to noncommutative geometry and the Ricci flow theory. This paradigm has several applications in physics, mainly string theory and gauge gravity. It might be meaningful to extend Vacaru's work for computing noncommutative Ricci flow quantum corrections with certain new applications in physics. Another motivating direction concerns the fractional formulation of Ricci flow theory which was constructed as well by Vacaru in [32]. In fact, fractional field theory is a new successful branch of theoretical physics characterized by fractional dimensions and with fractional differential operators [4, 15, 14, 11, 13, 12, 10, 18, 23, 31]. One main outcome resulting from Vacaru's approach is that a fractional Ricci flow can be considered as a nonholonomic evolution model deforming the Riemann geometry characterized by integer metrics and connections, symmetric and commutative spaces respectively to fractional metrics and connections, nonsymmetric and noncommutative spaces. It will be as well motivating to extend Vacaru's work on fractional and noncommutative Ricci flows theory [33] to include quantum corrections as well. Work in this direction is under progress.

**Acknowledgments.** I would like to thank the Key Laboratory of Numerical Simulation of Sichuan Province and the College of Mathematics of Neijiang Normal University for their financial support. I would like as well to express all my gratitude to Professor G.-C. Wu for his support and compassion. Much appreciations as well go for the anonymous referees.

#### REFERENCES

- [1] Caianiello, E. R., Feoli, A., Gasperini, M., Scarpetta G., *Quantum corrections to the spacetime metric from geometric phase space quantization*, Int. J. Theor. Phys. **29** (2) (1990), 131–139.

- 
- [2] Cao, H.-D., *Existence of gradient Kahler–Ricci solitons*, Elliptic and Parabolic Methods in Geometry (Minneapolis, 1994), 1–16, A. K. Peters, Wellesley MA, 1996.
- [3] Cao, H.-D., Zhu, X.-P., *A complete proof of the Poincare and Geometrization conjectures – application of the Hamilton–Perelman theory of the Ricci flow*, Asian J. Math., **10**, no. 2 (2006), 165–492.
- [4] Calcagni, G., *Geometry and field theory in multi-fractional spacetime*, JHEP01 (2012) 065, 61 pp.
- [5] D’Hoker, E., *String Theory, in Quantum Fields and Strings: A Course for Mathematicians*, vol. 2, American Mathematical Society, Providence, 1999.
- [6] Davis, S., Luckock, H., *The effect of higher-order curvature terms on string quantum cosmology*, Phys. Lett. B **485** (2000), 408–421.
- [7] Dowker, H. F., *Topology change in quantum gravity*, The Future of Theoretical Physics and Cosmology, eds. G. W. Gibbons, S. J. Rankin, E. P. S. Shellard, Cambridge Univ. Press, (2003), p. 879.
- [8] Dzhunushaliev, V., *Quantum wormhole as a Ricci flow*, Int. J. Geom. Meth. Mod. Phys. **6** (2009), 1033–1046.
- [9] Dzhunushaliev, V., Serikbayev, N., Myrzakulov, R., *Topology change in quantum gravity and Ricci flows*, arXiv:0912.5326v2.
- [10] El-Nabulsi, R. A., *Fractional field theories from multidimensional fractional variational problems*, Int. J. Mod. Geom. Meth. Mod. Phys. **5**, no. 6 (2008), 863–892.
- [11] El-Nabulsi, R. A., *Complexified quantum field theory and mass without mass from multidimensional fractional actionlike variational approach with time-dependent fractional exponent*, Chaos, Solitons Fractals **42**, no. 4 (2009), 2384–2398.
- [12] El-Nabulsi, R. A., *Fractional quantum field theory on multifractal sets*, American. J. Eng. Appl. Sci. **4** (1) (2010), 133–141.
- [13] El-Nabulsi, R. A., *Modifications at large distance from fractional and fractal arguments*, Fractals **18**, no. 2 (2010), 185–190.
- [14] El-Nabulsi, R. A., *Glaeske–Kilbas–Saigo fractional integration and fractional Dixmier trace*, Acta Math. Viet. **37**, no. 2 (2012), 149–160.
- [15] El-Nabulsi, R. A., Wu, G.-C., *Fractional complexified field theory from Saxena–Kumbhat fractional integral, fractional derivative of order and dynamical fractional integral exponent*, Afric. Disp. J. Math. **13**, no. 2 (2012), 45–61.
- [16] Gibbons, G. W., *Topology change in classical and quantum gravity*, Published in Mt. Sorak Symposium 1991: 159–185 (QCD161:S939:1991) Developments in Field Theory, ed. Jihn E Kim (Min Eum Sa, Seoul) (1992); arXiv:1110.0611v1.
- [17] Gibbons, G. W., Hawking, S. W., *Action integrals and partition functions in quantum gravity*, Phys. Rev. D **15** (1977), 2752–2756.
- [18] Goldfain, E., *Fractional dynamics and the standard model for particle physics*, Comm. Nonlinear Sci. Numer. Simul. **13** (2008), 1397–1404.
- [19] Hamilton, R., *Three manifolds with positive Ricci curvature*, Jour. Diff. Geom. **17** (1982), 255–306.
- [20] Hamilton, R., *Four-manifolds with positive curvature operator*, J. Diff. Geom. **24** (2) (1986), 153–179.
- [21] Hamilton, R., *Formation of singularities in the Ricci flow*, Surveys in Diff. Geom. **2** (1997), 7–136.
- [22] Headrick, M., Wiseman, T., *Ricci flow and black holes*, Class. Quant. Grav. **23** (2006), 6683–6708.
- [23] Herrmann, R., *Gauge invariance in fractional field theories*, Phys. Lett. A **372** (2008), 5515–5522.
- [24] Jolany, H., *Perelman’s functional and reduced volume*, arXiv:1004.1785.



- 
- [25] Mohaupt, T., *Strings, higher curvature corrections, and black holes*, talk given at the 2nd Workshop on Mathematical and Physical Aspects of Quantum Gravity, Blaubeuren, 28 July–1 August, (2005); hep-th/0512048.
- [26] Perelman, G., *Finite extinction time for the solutions to the Ricci flow on certain three manifolds*, math.DG/0307245v1.
- [27] Perelman, G., *Ricci flow with surgery on three-manifolds*, math.DG/0303109v1.
- [28] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, <http://arXiv.org/abs/math.DG/0211159>.
- [29] Thurston, William P., *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, American Mathematical Society. Bulletin. New Series **6** (3) (1982), 357–381.
- [30] Vacaru, S., *Spectral functionals, nonholonomic Dirac operators, and noncommutative Ricci flows*, J. Math. Phys. **50** (2009), 073503, 24 pp.
- [31] Vacaru, S., *Fractional dynamics from Einstein gravity, general solutions, and black holes*, Int. J. Theor. Phys. **51** (2012), 1338–1359.
- [32] Vacaru, S., *Fractional nonholonomic Ricci flows*, Chaos, Solitons and Fractals **45** (2012), 1266–1276.
- [33] Vacaru, S., *Nonholonomic Clifford and Finsler structures, non-commutative Ricci flows, and mathematical relativity*, arXiv: 1205.5387.

Rami Ahmad El-Nabulsi  
College of Mathematics and Information Science  
Neijiang Normal University  
Neijiang, Sichuan 641112  
China  
e-mail: nabulsiyahmadrami@yahoo.fr

Received October 17, 2011