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Generalization of *p*-regularity notion and tangent cone description in the singular case

Abstract. The theory of p-regularity has approximately twenty-five years' history and many results have been obtained up to now. The main result of this theory is description of tangent cone to zero set in singular case. However there are numerous nonlinear objects for which the p-regularity condition fails, especially for $p > 2$. In this paper we generalize the p-regularity notion as a starting point for more detailed consideration based on different p-factor operators constructions.

1. Introduction. In the setting of this article the local description of the solution set for curves and surfaces are essentially questions of p-regularity theory. In many classical cases these results can be viewed as the question about so-called singular points for the curve described by the equation

$$F(x, y) = 0,$$

where $(x, y) \in \mathbb{R}^2$.

The singular points (x_0, y_0) are such points for which the first partial derivatives are zeros, i.e.

$$\frac{\partial F}{\partial x}|_{(x_0,y_0)} = 0, \quad \frac{\partial F}{\partial y}|_{(x_0,y_0)} = 0.$$

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If additionally not all p-order partial derivatives equal 0 at points (x_0, y_0), then we say that such points are p-times irregular. For description of irregular points it is useful to investigate the sign of the determinant:

$$\Delta = \frac{\partial^2 F}{\partial x^2}|_{(x_0,y_0)} \cdot \frac{\partial^2 F}{\partial y^2}|_{(x_0,y_0)} - \frac{\partial^2 F}{\partial x \partial y}|_{(x_0,y_0)}.$$

Depending on the test of the sign of Δ we can classify and call irregular points. For $\Delta = 0$ the classification problems arise.

For this case we use the p-regularity theory where the basic apparatus is the so-called p-factor operator and p-factor method. The first result for description of zero set were obtained in [10], [11] for general case of p. It is necessary to point out that for $p = 1, 2$ the p-regularity notion is quite natural, but for $p \geq 3$ there are numerous nonlinear mappings such that p-regularity condition fails. For example, $F(x) = x_1 x_2^2$ or $F(x) = x_1 x_2^{p-1}$ and so on.

In this paper we generalize the p-regularity notion on much more nonlinear mappings and prove the theorem for description of the tangent cone to the zero set of the mappings in the singular case.

Finally we want to mention that p-factor method can be used to estimate the number of branch points with singularity of curves or surfaces, which we would like to realize in the present paper.

We begin with some notation. Throughout this paper we suppose that X and Y are Banach spaces. Let p be a natural number and let $B : X \times X \times \ldots \times X$ (p-copies of X)$\rightarrow Y$ be a continuous symmetric p-multilinear mapping. It means that B is defined on elements $x_1, x_2, \ldots, x_p \in X$ as $B(x_1, x_2, \ldots, x_p)$. The p form associated to B is the map $B[\cdot]^p : X \rightarrow Y$ defined by $B[x]^p = B(x, x, \ldots, x)$ for $x \in X$, where $x_1 = x_2 = \cdots = x_p = x$. Alternatively we may simply view $B[\cdot]^p$ as homogeneous polynomial $Q : X \rightarrow Y$ of degree p, i.e., $Q(\alpha x) = \alpha^p Q(x)$. About mapping F

$$F : X \rightarrow Y \tag{1.1}$$

we assume that it is $p+1$-times continuously differentiable on X and its pth order derivative at $x \in X$ will be denoted as $F^{(p)}(x)$ (a symmetric multilinear map $F^{(p)}(x)[h_1, h_2, \ldots, h_p]$ of p copies of X to Y). In a more detailed way, if $\mathcal{L}(X, Y)$ is the space of all linear operators from X to Y, then

$$\begin{aligned}
F'(x) &\in \mathcal{L}(X, Y), \\
F^{(2)}(x) &= (F'(x))' \in \mathcal{L}(X, \mathcal{L}(X, Y)), \\
&\vdots \\
F^{(p)}(x) &= (F^{(p-1)})'(x) \in \underbrace{\mathcal{L}(X, \mathcal{L}(X, \ldots, \mathcal{L}(X}_{p}, Y) \ldots)) = \mathcal{L}^p(X, Y).
\end{aligned}$$

The associated p-form, also called the pth order mapping, is

$$F^{(p)}(x)[h]^p = F^{(p)}(x)[\underbrace{h, h, \ldots, h}_{p}].$$

Moreover, the following necessary general formula holds:

$$F^{(p)}(x)[h_1 + h_2]^p = F^{(p)}(x)[h_1]^p + \tbinom{p}{1}F^{(p)}(x)[h_1]^{p-1}[h_2] + \tbinom{p}{2}F^{(p)}(x)[h_1]^{p-1}[h_2]^2 + \cdots + F^{(p)}(x)[h_2]^p.$$

Obviously from definition we have

$$F^{(p)}(x)[h_1]^q[h_2]^{p-q} = F^{(p)}(x)[\underbrace{h_1, \ldots, h_1}_{p}, \underbrace{h_2, \ldots, h_2}_{p-q}]. \tag{1.2}$$

For the mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, its derivatives $F^{(p)}(x)$ and their operations on elements h, we will use the notation as follows.

Let

$$F(x) = (f_1(x), \ldots, f_m(x))^T, \; x = [x_1, \ldots, x_n]^T,$$
$$h = [h_1, \ldots, h_n]^T, h^1 = [h_1^1, \ldots, h_n^1]^T, h^2 = [h_1^2, \ldots, h_n^2]^T.$$

Then

$$F'(x) = \left(\left(\frac{\partial f_1(x)}{\partial x_1}, \ldots, \frac{\partial f_1(x)}{\partial x_n}\right), \ldots, \left(\frac{\partial f_m(x)}{\partial x_1}, \ldots, \frac{\partial f_m(x)}{\partial x_n}\right)\right)^T,$$

$$F'(x)[h^1] = \left(\frac{\partial f_1(x)}{\partial x_1}h_1^1 + \ldots + \frac{\partial f_1(x)}{\partial x_n}h_n^1, \ldots, \frac{\partial f_m(x)}{\partial x_1}h_1^1 + \ldots + \frac{\partial f_m(x)}{\partial x_n}h_n^1\right)^T$$
$$= \left(\left(\frac{\partial}{\partial x_1}h_1^1 + \ldots + \frac{\partial}{\partial x_n}h_n^1\right) f_1(x), \ldots, \left(\frac{\partial}{\partial x_1}h_1^1 + \ldots + \frac{\partial}{\partial x_n}h_n^1\right) f_m(x)\right)^T,$$

$$F^{(2)}(x) = \left(\left(\left(\frac{\partial^2 f_1(x)}{\partial x_1 \partial x_1}, \ldots, \frac{\partial^2 f_1(x)}{\partial x_1 \partial x_n}\right), \ldots, \left(\frac{\partial^2 f_1(x)}{\partial x_n \partial x_1}, \ldots, \frac{\partial^2 f_1(x)}{\partial x_n \partial x_n}\right)\right), \ldots, \left(\left(\frac{\partial^2 f_m(x)}{\partial x_1 \partial x_1}, \ldots, \frac{\partial^2 f_m(x)}{\partial x_1 \partial x_n}\right), \ldots, \left(\frac{\partial^2 f_m(x)}{\partial x_n \partial x_1}, \ldots, \frac{\partial^2 f_m(x)}{\partial x_n \partial x_n}\right)\right)\right)^T,$$

$$F^{(2)}(x)[h^1] = \left(\left(\frac{\partial^2 f_1(x)}{\partial x_1 \partial x_1}h_1^1 + \cdots + \frac{\partial^2 f_1(x)}{\partial x_1 \partial x_n}h_n^1, \ldots, \frac{\partial^2 f_1(x)}{\partial x_n \partial x_1}h_1^1 + \ldots + \frac{\partial^2 f_1(x)}{\partial x_n \partial x_n}h_n^1\right), \ldots, \left(\frac{\partial^2 f_m(x)}{\partial x_1 \partial x_1}h_1^1 + \cdots + \frac{\partial^2 f_m(x)}{\partial x_1 \partial x_n}h_n^1, \ldots, \frac{\partial^2 f_m(x)}{\partial x_n \partial x_1}h_1^1 + \ldots + \frac{\partial^2 f_m(x)}{\partial x_n \partial x_n}h_n^1\right)\right)^T,$$

$$F^{(2)}(x)[h^1][h^2] = \Big(\frac{\partial^2 f_1(x)}{\partial x_1 \partial x_1} h_1^1 h_1^2 + \dots + \frac{\partial^2 f_1(x)}{\partial x_1 \partial x_n} h_n^1 h_1^2 + \dots$$
$$+ \frac{\partial^2 f_1(x)}{\partial x_n \partial x_1} h_1^1 h_n^2 + \dots + \frac{\partial^2 f_1(x)}{\partial x_n \partial x_n} h_n^1 h_n^2, \dots, \frac{\partial^2 f_m(x)}{\partial x_1 \partial x_1} h_1^1 h_1^2 + \dots$$
$$+ \frac{\partial^2 f_m(x)}{\partial x_1 \partial x_n} h_n^1 h_1^2 + \dots + \frac{\partial^2 f_m(x)}{\partial x_n \partial x_1} h_1^1 h_n^2 + \dots$$
$$+ \frac{\partial^2 f_m(x)}{\partial x_n \partial x_n} h_n^1 h_n^2\Big)^T$$

$$F^{(2)}(x)[h]^2 = \Big(\frac{\partial^2 f_1(x)}{\partial x_1 \partial x_1} h_1^2 + \dots + \frac{\partial^2 f_1(x)}{\partial x_1 \partial x_n} h_n h_1 + \dots + \frac{\partial^2 f_1(x)}{\partial x_n \partial x_1} h_1 h_n + \dots$$
$$+ \frac{\partial^2 f_1(x)}{\partial x_n \partial x_n} h_n^2, \dots, \frac{\partial^2 f_m(x)}{\partial x_1 \partial x_1} h_1^2 + \dots + \frac{\partial^2 f_m(x)}{\partial x_1 \partial x_n} h_n h_1 + \dots$$
$$+ \frac{\partial^2 f_m(x)}{\partial x_n \partial x_1} h_1 h_n + \dots + \frac{\partial^2 f_m(x)}{\partial x_n \partial x_n} h_n^2\Big)^T$$
$$= \bigg(\Big(\frac{\partial}{\partial x_1} h_1 + \dots + \frac{\partial}{\partial x_n} h_n\Big)^2 f_1(x), \dots,$$
$$\Big(\frac{\partial}{\partial x_1} h_1 + \dots + \frac{\partial}{\partial x_n} h_n\Big)^2 f_m(x)\bigg)^T$$

and generally

$$(F^{(p)}(x)[h]^p = \bigg(\Big(\frac{\partial}{\partial x_1} h_1 + \dots + \frac{\partial}{\partial x_n} h_n\Big)^p f_1(x), \dots, \Big(\frac{\partial}{\partial x_1} h_1 + \dots + \frac{\partial}{\partial x_n} h_n\Big)^p f_m(x)\bigg)^T. \tag{1.3}$$

Remark, that we can write

$$f_i^{(p)}(x)[h]^q = (f_i^{(p)}(x)[h]^{q-1})[h], \ i = 1, \dots, m, \tag{1.4}$$

where for $q = 1$ we have

$$f_i^{(p)}(x) = \underbrace{\Big(\Big(\dots\Big(}_{p} \underbrace{\frac{\partial^p f_i(x)}{\partial x_1 \dots \partial x_1 \partial x_1}}_{p}, \dots, \underbrace{\frac{\partial^p f_i(x)}{\partial x_1 \dots \partial x_1 \partial x_n}}_{p} \underbrace{\Big), \dots\Big)}_{p-1},$$
$$\dots, \underbrace{\Big(\dots, \Big(}_{p-1} \underbrace{\frac{\partial^p f_i(x)}{\partial x_n \dots \partial x_n \partial x_1}}_{p}, \dots, \underbrace{\frac{\partial^p f_i(x)}{\partial x_n \dots \partial x_n \partial x_n}}_{p} \underbrace{\Big) \dots\Big)\Big)}_{p}, \tag{1.5}$$

$$
(1.6) \quad
\begin{aligned}
f_i^{(p)}(x)[h] = &\underbrace{\Big(\dots\Big(}_{p-1}\underbrace{\frac{\partial^p f_i(x)}{\partial x_1 \dots \partial x_1 \partial x_1}}_{p} h_1 + \dots + \underbrace{\frac{\partial^p f_i(x)}{\partial x_1 \dots \partial x_1 \partial x_n}}_{p} h_n, \dots \Big), \\
&\dots, \\
&\Big(\dots, \underbrace{\frac{\partial^p f_i(x)}{\partial x_n \dots \partial x_n \partial x_1}}_{p} h_1 + \dots + \underbrace{\frac{\partial^p f_i(x)}{\partial x_n \dots \partial x_n \partial x_n}}_{p} h_n \underbrace{\Big) \dots \Big)}_{p-1}.
\end{aligned}
$$

Obviously

$$
f_i^{(p)}(x)[h^1, \dots, h^q] = f_i^{(p)}(x)[h^1, \dots, h^{q-1}][h^q],
$$
$$
i = 1, \dots, m, \quad h^r \in \mathbb{R}^n, \quad r = 1, \dots, q, \quad q = 2, \dots, p.
$$

We also assume that F is completely degenerate up to order p at the point x^*, i.e.

$$
F^{(i)}(x^*) = 0 \tag{1.7}
$$

for $i = 1, 2, \dots, p-1$.

Now we formulate the fundamental definitions and results for this case (see [5]).

Definition 1.1. Linear operator $\Psi_p(h) \in \mathcal{L}(X, Y)$, for some fixed $h \in X \setminus \{0\}$, defined by

$$
\Psi_p(h) : \xi \to \Psi_p(h)[\xi] = F^{(p)}(x^*)[h]^{p-1}[\xi], \qquad \xi \in X \tag{1.8}
$$

is called a p-factor operator.

Definition 1.2. The mapping

$$
F : X \to Y \tag{1.9}
$$

is called p-regular at a point x^* along element $h \in X$ if the condition

$$
\Psi_p(h)X = Y \;\text{ or }\; \mathrm{Im}\Psi_p(h) = Y \tag{1.10}
$$

holds.

From these notions it is implied that for any fixed element $h \in X$ there exists a linear operator $F^{(p)}(x^*)[h]^{p-1} \in \mathcal{L}(X, Y)$ such that for any $\xi \in X$

$$
F^{(p)}(x^*)[h]^{p-1}[\xi] = F^{(p)}(x^*)[\underbrace{h, \dots, h}_{p-1}, \xi]. \tag{1.11}
$$

In many of the most important applications are such situations when we consider the so-called p-kernel of p-derivatives. The p-kernel of $F^p(x^*)$ is the following set

$$
\mathrm{Ker}^p F^{(p)}(x^*) = \{h \in X : F^{(p)}(x^*)[h]^p = 0\}. \tag{1.12}
$$

It is shown by the two examples underneath.

Example 1.3. Consider the mapping

$$F : \mathbb{R}^2 \to \mathbb{R},$$

$$F(x) = F(x_1, x_2) = x_1^2 - x_2^2, \quad x^* = (0,0). \tag{1.13}$$

According to the following calculations

$$F'(x) = (2x_1, -2x_2), \; F'(x^*) = (0,0),$$

$$F^{(2)}(x) = ((2,0);(0,-2)) = F^{(2)}(x^*) \neq 0$$

the 2-factor operator has the form

$$F^{(2)}(x^*)[h] = (2h_1, -2h_2) \text{ for } h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \tag{1.14}$$

and

$$F^2(x^*)[h]^2 = 2h_1^2 - 2h_2^2.$$

Then we obtain the 2-kernel of the operator of the second derivative at the point x^* as follows

$$\mathrm{Ker}^2 F^{(2)}(x^*) = lin\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \cup lin\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} \tag{1.15}$$

and the 2-factor operators Ψ_2

$$\Psi_2(x^*, \overline{h}) = F^{(2)}(x^*)\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = (2,-2), \tag{1.16}$$

$$\Psi_2(x^*, \overline{\overline{h}}) = F^{(2)}(x^*)\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] = (2,2) \tag{1.17}$$

where $\overline{h} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, $\overline{\overline{h}} = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$.

The image of the first 2-factor operator has the form

$$\begin{aligned} \mathrm{Im}\Psi_2(x^*, \overline{h}) &= \left\{x \in \mathbb{R} : x = \Psi_2(x^*, \overline{h})\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\} \\ &= \{x \in \mathbb{R} : x = 2y_1 - 2y_2 = 2(y_1 - y_2)\} = \mathbb{R}. \end{aligned} \tag{1.18}$$

Analogously

$$\begin{aligned} \mathrm{Im}\Psi_2(x^*, \overline{\overline{h}}) &= \left\{x \in \mathbb{R} : x = \Psi_2(x^*, \overline{\overline{h}})\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\} \\ &= \{x \in \mathbb{R} : x = 2y_1 + 2y_2 = 2(y_1 + y_2)\} = \mathbb{R}. \end{aligned} \tag{1.19}$$

Therefore the mapping F is 2-regular at the point $x^* = 0$ along the elements of the 2-kernel of the second derivative.

Example 1.4. As a map F we take

$$F:\mathbb{R}^2 \to \mathbb{R}, \tag{1.20}$$

$$F(x,y) = (x^2+y^2)^2 - a^2(x^2-y^2), \qquad a \neq 0. \tag{1.21}$$

It is well known that equation $F(x,y) = 0$ represents lemniscate of Bernoulli (treated as foot lines of equiaxial hyperbolas). We only have three points x^*:

$$x^* = (0,0),(a,0),(-a,0) \tag{1.22}$$

as solutions of the equation $F(x) = 0$ but only one of them $x^* = (0,0)$ is interesting for our investigations.

The derivation of the map F gives us

$$F'(x) = (4x^3 + 4xy^2 - 2a^2x; 4x^2y + 4y^3 + 2a^2y) \tag{1.23}$$

and for $x^* = (0,0)$ we have in (1.7) $p = 2$ and

$$F'(0,0) = 0.$$

For two another points (1.22) we get $F'(x^*) \neq 0$.

Then the singularity exists only at the point $x^* = (0,0)$. The second derivative of $F(x)$ gives us

$$F^{(2)}(x) = ((12x^2 + 4y^2 - 2a^2, 8xy); (8xy, 4x^2 + 12y^2 + 2a^2)) \tag{1.24}$$

and at $x^* = 0$ we get

$$F^{(2)}(0,0) = ((-2a^2, 0); (0, 2a^2)),$$

thus $F^{(2)}(x^*) \neq 0$, $a \neq 0$.

Next the 2-factor operator has the following form

$$F^{(2)}(x^*)[h] = (-2a^2h_1, 2a^2h_2) \tag{1.25}$$

and

$$F^{(2)}(x^*)[h]^2 = -2a^2(h_1^2 - h_2^2). \tag{1.26}$$

Carrying through the calculations of the 2-kernel of $F^{(2)}(x^*)$ as follows

$$\mathrm{Ker}^2 F^{(2)}(x^*) = lin\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} \cup lin\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}, \tag{1.27}$$

we obtain the linear 2-factor operators Ψ_2

$$\Psi_2(x^*,\overline{h}) = F^{(2)}(x^*)\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = (-2a^2, 2a^2) \tag{1.28}$$

$$\Psi_2(x^*,\overline{\overline{h}}) = F^{(2)}(x^*)\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right] = (-2a^2, -2a^2) \tag{1.29}$$

for which the images state

$$\mathrm{Im}\Psi_2(x^*,\overline{h}) = \mathrm{Im}\Psi_2(x^*,\overline{\overline{h}}) = \mathbb{R}. \tag{1.30}$$

For example for (1.28) we have

$$\operatorname{Im}\Psi_2(x^*, h_1) = \left\{x \in \mathbb{R} : x = \Psi_2(x^*, \overline{h})\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\}$$
$$= \left\{x \in \mathbb{R} : x = -2a^2 y_1 + 2a^2 y_2 = -2a^2(y_1 - y_2)\right\} = \mathbb{R}.$$

It means that the mapping F is 2-regular at the point $x^* = 0$ along the elements of the 2-kernel of the second derivative.

In many nonlinear operator equations and geometric theories we can describe the solution set of mappings F with the help of the so-called tangent cone to the level set of the mapping F. Denote this solution set by

$$M(x^*) = \{x \in U : F(x) = F(x^*)\}. \tag{1.31}$$

Definition 1.5. An element $h \in X$ is called tangent vector at a point $x^* \in X$ to some nonempty set $M(x^*)$ if there exists a number $\varepsilon > 0$ and a mapping:

$$r : [0, \varepsilon] \to X, \tag{1.32}$$

such that

$$\forall (t \in [0, \varepsilon])\, x^* + th + r(t) \in M(x^*), \tag{1.33}$$

where

$$\|r(t)\| = o(t) \quad \text{for} \quad t \to 0. \tag{1.34}$$

We can also say that the element h is a p-tangent vector to the set $M(x^*)$ at the point x^* if

$$\|r(t)\| = o(t^p),\ p \geq 1.$$

The set of all p-tangent vectors at the point $x^* \in M(x^*)$ is called a p-tangent cone, and denoted as $T_pM(x^*)$.

Now we are ready to generalize our considerations.

Theorem 1.6 (Tret'yakov – generalization of Lyusternik theorem). *Let F be the mapping: $F : X \to Y$, $F \in C^{p+1}(X)$ and*

$$F^{(i)}(x^*) = 0, \qquad i = 1, \ldots, p-1.$$

If the mapping F is p-regular at the point x^* along all elements $h \in \operatorname{Ker}^p F^{(p)}(x^*)$, then

$$T_1 M(x^*) = \operatorname{Ker}^p F^{(p)}(x^*).$$

Applying this theorem to Example 1.4, we can show that in the 2-regular case the tangent cone $T_1M(x^*) = \operatorname{Ker}^2 F^{(2)}(x^*)$ is always two-sided. It means that there exist two tangent lines at point the x^*. They are $y = x$ and $y = -x$.

We conclude this section with a useful lemma which we need for our further consideration. Let $\rho(x, y) = \|x - y\|$ be a distance between elements

x and y in Banach space and $\rho(x, M) = \inf\{\|x - z\| : z \in M\}$ be the distance of element x from the subset M in this space. By $dist_H(A_1, A_2)$ we mean the Hausdorff distance between sets A_1 and A_2.

Lemma 1.7 (Contraction multimapping principle ([3], [8])). *Let Z be a Banach space. Assume that we are given a multimapping*

$$\Phi : U_\varepsilon(z_0) \to 2^Z$$

on a ball $U_\varepsilon(z_0) = \{z : \rho(z, z_0) < \varepsilon\}$ ($\varepsilon > 0$), where the sets $\Psi(z)$ are non-empty and closed for any $z \in U_\varepsilon(z_0)$. Further, assume that there exists a number θ, $0 < \theta < 1$, such that

1. *$dist_H(\Phi(z_1), \Phi(z_2)) \le \theta\rho(z_1, z_2)$ for any $z_1, z_2 \in U_\varepsilon(z_0)$,*
2. *$\rho(z_0, \Phi(z_0)) < (1 - \theta)\varepsilon$.*

Then, for every number ε_1 which satisfies the inequality

$$\rho(z_0, \Phi(z_0)) < \varepsilon_1 < (1 - \theta)\varepsilon,$$

there exists $z \in B_{\varepsilon_1/(1-\theta)} = \{\omega : \rho(\omega, z_0) \le \varepsilon_1/(1 - \theta)\}$ such that

$$z \in \Phi(z).$$

2. Generalization of *p*-regularity and description of tangent cones. In this part we consider a generalization of the concept of p-regularity given in [1], [2]. The new necessary optimality conditions for extremum problems with singularities (1.7) was derived by O. Brezhneva and A. A. Tret'yakov. They described the tangent cone $T_1M(x^*)$ to solution set $M(x^*)$ in singular case (1.7) under an assumption that the mapping F does not satisfy the condition of p-regularity given in Definition 1.2. Under our generalization of p-regularity notion we describe the tangent cones of a new class of mappings. We start from theorem proved in [1] in a modified form.

Let us denote $Y_1(h) = \mathrm{Im}F^{(p)}(x^*)[h]^{p-1}$ and $Y_2(h)$ is a linear subspace that complements $Y_1(h)$ with respect to Y and both are closed in Y and let $P_2(h)$ be the projection operator onto $Y_2(h)$ along $Y_1(h)$.

Theorem 2.1. *Let $F : X \to Y$, X and Y are Banach spaces, $F \in C^{p+1}(X)$, $F(x^*) = 0$. Suppose that*

$$Y = Y_1(h) \oplus Y_2(h) \tag{2.1}$$

is satisfied for

$$h \in \mathrm{Ker}^p F^{(p)}(x^*),$$

where $F^{(k)}(x^*) = 0$, $k = 1, \ldots, p-1$ and there exists an element $\widetilde{h} \in X$, ($c \le \|h\|$, $\left\|\widetilde{h}\right\| < C$, $0 < c \le C < \infty$ and $D > 0$) such that

$$\left\|F^{(p)}(x^*)[h]^{p-1}[\widetilde{h}]\right\| = 0 \tag{2.2}$$

and

$$\left\|P_2(h)F(x^* + th + t^\alpha\widetilde{h})\right\| \le t^{(p-2)+2+\alpha} \tag{2.3}$$

for some $1 < \alpha \le \frac{3}{2}$, $\varepsilon \in (0, 1)$,

$$\left\| \left(F^{(p)}(x^*)[h]^{p-1} + P_2(h) F^{(p)}(x^*)[h]^{p-2}[\widetilde{h}] \right)^{-1} \right\| \le D, \tag{2.4}$$

where $t \in (0, \delta)$ *and* $\delta > 0$ *is sufficiently small.*

Then

$$h \in T_1 M(x^*).$$

Moreover,

$$\alpha^+(t) = x^* + th + t^\alpha \widetilde{h} + r^+(t) \in M(x^*)$$
$$\alpha^-(t) = x^* + th - t^\alpha \widetilde{h} + r^-(t) \in M(x^*),$$

where

$$\|r^\pm(t)\| = o(t^\alpha).$$

As a consequence of Theorem 2.1 we obtain the following corollary which is very important and useful for our next investigations for $p = 4$.

Corollary 2.2. *Let the point* $x^* \in X$ *and* F *be mapping such that* $F \in C^5(X, Y)$, $F^{(k)}(x^*) = 0$ *for* $k = 1, 2, 3$. *Suppose that* $h \in \mathrm{Ker}^4 F^{(4)}(x^*)$, $h \neq 0$, $F^{(4)}(x^*)[h]^3 = 0$ *and there exist an element* $\widetilde{h} \in X$, $\widetilde{h} \neq 0$ *and a number* $c > 0$ *such that*

$$\left\| F(x^* + th + t^\alpha \widetilde{h}) \right\| \le t^{2+2\alpha+\varepsilon}, \quad \varepsilon \in (0, 1), \quad \textit{for some } 1 < \alpha \le 2, \tag{2.5}$$

$$\left\| \{ F^{(4)}(x^*)[h]^2[\widetilde{h}] \}^{-1} \right\| \le c \tag{2.6}$$

$t \in (0, \delta)$, *where* $\delta > 0$ *is sufficiently small.*

Then

$$h \in T_1 M(x^*).$$

Moreover,

$$\gamma^+(t) = x^* + th + t^\alpha \widetilde{h} \pm r^+(t) \in M(x^*) \tag{2.7}$$

$$\gamma^-(t) = x^* + th - t^\alpha \widetilde{h} \pm r^-(t) \in M(x^*), \tag{2.8}$$

where

$$\|r^\pm(t)\| = o(t^\alpha).$$

Consider some examples which illustrate this result.

Example 2.3. Let us consider the rose-petal curve called quadrofolium described by the equation:

$$F(x, y) = (x^2 + y^2)^3 - 4a^2 x^2 y^2 = 0, \quad a \neq 0. \tag{2.9}$$

We show that all conditions of corollary are fulfilled. Simple calculations for the singularity point $x^* = (0, 0)$ give us the following:

$$F'(x) = (6x^5 + 12x^3y^2 + 6xy^4 - 8a^2xy^2; 6x^4y + 12x^2y^3 + 6y^5 - 8a^2x^2y) \tag{2.10}$$

$$F'(x^*) = (0,0) = 0.$$

One can verify that $F^{(2)}(x^*) = 0$, $F^{(3)}(x^*) = 0$ and the forth derivative at $x^* = (0,0)$ is

$$\begin{aligned} F^{(4)}(x^*) = ((((0,0),(0,-16a^2));((0,-16a^2),(-16a^2,0)));\\ (((0,-16a^2),(-16a^2,0));((-16a^2,0),(0,0)))) \end{aligned} \tag{2.11}$$

and it means that $F^{(4)}(x^*) \neq 0$. After simple calculations and applying the specifications to the 4-kernel of $F^{(4)}(x^*)$ as follows

$$\mathrm{Ker}^4 F^{(4)}(x^*) = lin\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \cup lin\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

we obtain after verification for both elements

$$F^{(4)}(x^*)[h] \neq 0, \quad F^{(4)}(x^*)[h]^2 \neq 0, \quad F^{(4)}(x^*)[h]^3 = 0, \quad F^{(4)}(x^*)[h]^4 = 0.$$

According to the corollary there exists an element $\widetilde{h} \neq 0$ such that conditions (2.5)–(2.6) are fulfilled and it means that there exist the curves $\gamma^+(t)$ and $\gamma^-(t)$ ((2.7) and (2.8) respectively). So we can take $h = \begin{bmatrix}1\\0\end{bmatrix}$ and $\widetilde{h} = \begin{bmatrix}0\\1\end{bmatrix}$ such that $F^{(4)}(x^*)\begin{bmatrix}1\\0\end{bmatrix}^3\begin{bmatrix}0\\1\end{bmatrix} = 0$, $F^{(4)}(x^*)\begin{bmatrix}1\\0\end{bmatrix}^2\begin{bmatrix}0\\1\end{bmatrix} = -16a^2 \neq 0$. Thus (2.6) is fulfilled and we need only to verify condition (2.5). We obtain for $\widetilde{h} = \begin{bmatrix}0\\\beta\end{bmatrix}$

$$(t^2 + (\beta t)^{2\alpha})^3 - 4a^2t^2((\beta t)^\alpha)^{2\alpha} = t^6 - 4a^2t^{2\alpha}\beta^2t^4 + o(t^{2+2\alpha+\varepsilon}) = 0.$$

It means that for $\beta = \frac{1}{2a}$ and $\alpha = 2$ we have

$$\|F(t,(\beta t)^\alpha)\| = 3t^8\beta^2 + 3t^{10}\beta^4 + t^{12}\beta^6 \leq t^{6+\varepsilon}.$$

All conditions are fulfilled for sufficiently small $\varepsilon > 0$, then

$$h = \begin{bmatrix}1\\0\end{bmatrix} \in T_1M(x^*).$$

In a similar way we can show that $h = \begin{bmatrix}0\\1\end{bmatrix} \in T_1M(x^*)$.

Theorem 2.1 can be applied to the next example as well.

Example 2.4. Cardioid with one cusp can be described as follows:

$$F(x,y) = \left(x^2 + y^2 - \frac{1}{2}ax\right)^2 - \frac{1}{2}a^2(x^2+y^2) = 0, \quad a \neq 0.$$

From the simple calculations we obtain:

$$F'(x^*) = 0, \quad F^{(2)}(x^*) \neq 0, \; F^{(2)}(x^*)[h] = \left(0, -\frac{1}{2}a^2h_2\right),$$

$$F^{(2)}(x^*)[h]^2 = -\frac{1}{2}a^2 h_2^2,$$

$\mathrm{Ker}^2 F^{(2)}(x^*) = lin\left\{\binom{1}{0}\right\}$. For $h = \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}$ we find $\widetilde{h} = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$

$$\left\| F(th + t^\alpha \widetilde{h}) \right\| \le t^{2\alpha+\varepsilon}, \alpha = \frac{3}{2}$$

and we notice that all conditions of Theorem 2.1 are fulfilled, which implies that $x^* + th + t^\alpha \widetilde{h} + r^+(t) \in M(x^*)$, where

$$\left\| r^+(t) \right\| = o(t^\alpha).$$

It means that $h \in T_1 M(x^*)$. But for $h = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathrm{Ker}^2 F^{(2)}(x^*)$ condition (2.3) does not hold. And it is obviously $h = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin T_1 M(x^*)$. The tangent cone $T_1 M(x^*)$ is not two-sided.

The following theorem is the main result of this paper and describes just such a situation for $p = 3$.

Theorem 2.5. *Let* $F : \mathbb{R}^n \to \mathbb{R}^m$, $F \in C^4(\mathbb{R}^n, \mathbb{R}^m)$ *and* $F(x^*) = 0$, $F'(x^*) = 0$, $F^{(2)}(x^*) = 0$. *Additionally, let* $h \in Ker^3 F^{(3)}(x^*)$ *and*

$$F^{(3)}(x^*)[h]^2 = 0. \tag{2.12}$$

If there exists some $\overline{h} \neq 0$ *such that*

$$\mathrm{Im}(F^{(3)}(x^*)[h, \overline{h}]) = \mathbb{R}^m, \tag{2.13}$$

$$F^{(3)}(x^*)[h][\overline{h}]^2 = 0, \tag{2.14}$$

then

$$h \in T_1 M(x^*).$$

Proof. Let $x(\alpha) = x^* + \alpha h + \alpha^{\frac{5}{4}} \overline{h}$, $\Lambda(\alpha) = F^{(3)}(x^*)\left[\alpha h, \alpha^{\frac{5}{4}} \overline{h}\right]$ and $\Lambda = F^{(3)}[h, \overline{h}]$. We will prove that $h \in T_1 M(x^*)$. To this end it is sufficient to show that there exists $r(\alpha)$ such that $F(x^* + \alpha h + \alpha^{\frac{5}{4}} \overline{h} + r(\alpha)) = 0$ and $\|r(\alpha)\| = o(\alpha^{\frac{5}{4}})$.

For this purpose we construct the following sequence

$$x_{k+1} = x_k - \{F^{(3)}(x^*)[\alpha h, \alpha^{\frac{5}{4}} \overline{h}]\}^{-1} F(x^* + \alpha h + \alpha^{\frac{5}{4}} \overline{h} + x_k), \quad x_0 = 0 \tag{2.15}$$

and consider the multivalued mapping

$$\Psi(x) = x - \{F^{(3)}(x^*)[\alpha h, \alpha^{\frac{5}{4}} \overline{h}]\}^{-1} F(x^* + \alpha h + \alpha^{\frac{5}{4}} \overline{h} + x), \tag{2.16}$$

where

$$\{F^{(3)}[\alpha h, \alpha^{\frac{5}{4}} \overline{h}]\}^{-1} \stackrel{df}{=} \Lambda^{-1}(\alpha)$$

is the right-inverse operator ([3]).

From the Banach open mapping theorem it follows that

$$\left\|\Lambda^{-1}(\alpha)y\right\| \leq \frac{c}{\alpha^{\frac{9}{4}}}\,\|y\| = c(\Lambda)\,\|y\|\,, \quad c(\Lambda) \overset{df}{=} \frac{\overline{c}}{\alpha^{\frac{9}{4}}}. \tag{2.17}$$

We will prove that the operator $\Psi(x)$ described in (2.16) has a fixed point $r(\alpha)$ such that $r(\alpha) \in \Psi(r(\alpha))$ and $\|r(\alpha)\| = o(\alpha^{\frac{5}{4}})$. For this purpose we need to show that $\Psi(x)$ is a contraction in some neighborhood $U(0) = \{x \in X : \|x\| \leq c\alpha^{\frac{5}{4}+\varepsilon}\}$ (with respect to the Hausdorff distance), where $c > 0$ is an independent constant and $\|\Psi(0)\| = o(\alpha^{\frac{5}{4}})$. It means that the Hausdorff distance

$$dist_H(\Psi(x_1), \Psi(x_2)) \leq \Theta\|x_1 - x_2\| \tag{2.18}$$

for $\Theta \in (0, 1)$ and sufficiently small $\alpha > 0$. Taking $x_1, x_2 \in U(0)$ such that

$$\|x_1\| \leq c\alpha^{\frac{5}{4}+\varepsilon}, \|x_2\| \leq c\alpha^{\frac{5}{4}+\varepsilon} \tag{2.19}$$

we have

$$\Psi(x_1) = x_1 - \Lambda^{-1}(\alpha)F(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_1) \tag{2.20}$$

and

$$\Psi(x_2) = x_2 - \Lambda^{-1}(\alpha)F(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_2) \tag{2.21}$$

and according to (2.17) the following estimation for distance holds

$$\begin{aligned} dist_H(\Psi(x_1), \Psi(x_2)) \leq c(\Lambda)\,\Big\|\Lambda(x_1 - x_2) - F(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_1) \\ - F(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_2)\Big\|, \end{aligned} \tag{2.22}$$

where $c(\Lambda)$ is Banach's constant depending on α. Using the "mean value" theorem, we get

$$\begin{aligned} &dist_H(\Psi(x_1), \Psi(x_2)) \\ &\leq c(\Lambda(\alpha))\,\left\|\Lambda(\alpha) - F'(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_1 + \Delta)\right\|\,\|x_1 - x_2\|\,, \end{aligned} \tag{2.23}$$

where

$$\|\Delta\| \leq \overline{c}\alpha^{\frac{5}{4}+\varepsilon}$$

and applying Taylor's series, we obtain

$$\begin{aligned} F'(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_1 + \Delta) &= F'(x^*) + F^{(2)}(x^*)[\alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_1 + \Delta] \\ &+ \frac{1}{2}F^{(3)}(x^*)[\alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_1 + \Delta]^2 + \gamma(\alpha), \end{aligned}$$

where $\|\gamma(\alpha)\| = o(\alpha^3)$.

According to the conditions of our theorem we get

$$F^{(2)}(x^*)[\alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_1 + \Delta] = 0$$

and then

$$\begin{aligned} &F'(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_1 + \Delta) \\ &\quad = \frac{1}{2}F^{(3)}(x^*)[\alpha h]^2 + F^{(3)}(x^*)[\alpha h, \alpha^{\frac{5}{4}}\overline{h}] + \frac{1}{2}F^{(3)}(x^*)[\alpha^{\frac{5}{4}}\overline{h}]^2 + \omega(\alpha) \\ &\quad = F^{(3)}(x^*)[\alpha h, \alpha^{\frac{5}{4}}\overline{h}] + \frac{1}{2}F^{(3)}(x^*)[\alpha^{\frac{5}{4}}\overline{h}]^2 + \omega(\alpha), \end{aligned}$$

where $\omega(\alpha)$ is such that

$$\|\omega(\alpha)\| \le c\alpha^{\frac{5}{4}+\varepsilon+1} = \alpha^{\frac{9}{4}+\varepsilon}.$$

We substitute the last expression to (2.22), keeping (2.23) in mind. We obtain

$$\begin{aligned} &\Lambda(\alpha) - F'(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_1 + \Delta) \\ &\quad = F^{(3)}(x^*)[\alpha h, \alpha^{\frac{5}{4}}\overline{h}] - F^{(3)}(x^*)[\alpha h, \alpha^{\frac{5}{4}}\overline{h}] - \frac{1}{2}F^{(3)}(x^*)[\alpha^{\frac{5}{4}}\overline{h}]^2 - \omega(\alpha) \\ &\quad = -\frac{1}{2}F^{(3)}(x^*)[\alpha^{\frac{5}{4}}\overline{h}]^2 - \omega(\alpha). \end{aligned}$$

It means that

$$\left\|\Lambda(\alpha) - F'(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h} + x_1 + \Delta)\right\| \le c(\alpha^{\frac{10}{4}} + \alpha^{\frac{9}{4}+\varepsilon}) \le 2c\alpha^{\frac{9}{4}+\varepsilon},$$

where $\varepsilon > 0$ is sufficiently small. Thus in (2.22) we obtain

$$dist_H(\Psi(x_1), \Psi(x_2)) \le c(\Lambda(\alpha)) \|x_1 - x_2\| \, 2c\alpha^{\varepsilon+\frac{3}{4}} \le c^2\alpha^{\varepsilon} \|x_1 - x_2\| . \tag{2.24}$$

For sufficiently small α it gives

$$dist_H(\Psi(x_1), \Psi(x_2)) \le \Theta \|x_1 - x_2\| ,$$

where $\Theta = 2c^2\alpha^{\varepsilon} < 1$. The last inequality proves contraction condition for the operator $\Psi(x)$.

Next we will show that $\Psi(0) = o(\alpha^{\frac{5}{4}})$. This condition holds if

$$\|x_1\| \le c\alpha^{\frac{5}{4}+\varepsilon}, \tag{2.25}$$

where

$$x_1 = \{F^{(3)}(x^*)[\alpha h, \alpha^{\frac{5}{4}}\overline{h}]\}^{-1} F(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h}).$$

First one ought to notice that

$$\begin{aligned} \|x_1\| &\le \left\|\Lambda^{-1}(\alpha)\right\| \left\|F(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h})\right\| \\ &\le \frac{c}{\alpha^{\frac{9}{4}}} \Big\| F(x^*) - F'(x^*)\left[\alpha h + \alpha^{\frac{5}{4}}\overline{h}\right] \\ &+ \frac{1}{2}F^{(2)}(x^*)\left[\alpha h + \alpha^{\frac{5}{4}}\overline{h}\right]^2 + \frac{1}{6}F^{(3)}(x^*)\left[\alpha h + \alpha^{\frac{5}{4}}\overline{h}\right]^3 \Big\| + \omega(\alpha) \end{aligned}$$

and

$$\|\omega(\alpha)\| \le c\alpha^4.$$

From conditions of our theorem we have

$$F(x^*) = F'(x^*) = F^{(2)}(x^*) = 0$$

and

$$\begin{aligned}
\|x_1\| &\le \frac{c}{6\alpha^{\frac{9}{4}}} \left\| F^{(3)}(x^*) \left[\alpha h + \alpha^{\frac{5}{4}}\overline{h}\right]^3 \right\| + o(\alpha^4) \\
&= \frac{c}{6\alpha^{\frac{9}{4}}} \Big\| F^{(3)}(x^*) [\alpha h]^3 + 3F^{(3)}(x^*) [\alpha h]^2 \left[\alpha^{\frac{5}{4}}\overline{h}\right] \\
&\quad + 3F^{(3)}(x^*) [\alpha h] \left[\alpha^{\frac{5}{4}}\overline{h}\right]^2 + F^3(x^*) \left[\alpha^{\frac{5}{4}}\overline{h}\right]^3 \Big\| + o(\alpha^4) \\
&= \frac{c}{6\alpha^{\frac{9}{4}}} \Big\| 3\alpha^2\alpha^{\frac{5}{4}} F^{(3)}(x^*) [h]^2 \left[\overline{h}\right] + 3\alpha\alpha^3 F^{(3)}(x^*) [h] \left[\overline{h}\right]^2 \\
&\quad + \alpha^{\frac{15}{4}} F^{(3)}(x^*) \left[\overline{h}\right]^3 \Big\| + o(\alpha^4)
\end{aligned}$$

and from (2.14) we obtain

$$\|x_1\| \le \frac{c}{6\alpha^{\frac{9}{4}}} \left\| \alpha^{\frac{15}{4}} F^3(x^*) \left[\overline{h}\right]^3 \right\| + o(\alpha^4) \le \frac{c}{\alpha^{\frac{9}{4}}} \left(\alpha^{\frac{15}{4}} + \alpha^4\right) \le \alpha^{\frac{6}{4}} + \alpha^{\frac{7}{4}} = \alpha^{\frac{5}{4}+\varepsilon}.$$

It means that (2.25) is fulfilled and $\|\Psi(0)\| = o(\alpha^{\frac{5}{4}})$.

The generalized contraction mapping principle implies that there exists $r(\alpha)$ such that $r(\alpha) \in \Psi(r(\alpha))$ and $\|r(\alpha)\| = o(\alpha^{\frac{5}{4}})$. It gives $F(x^* + \alpha h + \alpha^{\frac{5}{4}}\overline{h} + r(\alpha)) = 0$. This means that $h \in T_1 M(x^*)$ and the proof is complete. □

We illustrate this theorem in

Example 2.6. Let us consider the surface called foot surface given by the equation:

$$F(x, y, z) = 27a^3xyz - (x^2 + y^2 + z^2)^3 = 0. \tag{2.26}$$

For $x^* = (0, 0, 0)$ we have

$$F(x^*) = 0,\ F'(x^*) = 0,\ F''(x^*) = 0.$$

The third derivative at the point x^* has the form:

$$\begin{aligned}
F^{(3)}(x^*) = &(((0, 0, 0), (0, 0, -27a^3), (0, -27a^3, 0)); \\
&((0, 0, -27a^3), (0, 0, 0), (-27a^3, 0, 0)); \\
&((0, -27a^3, 0), (-27a^3, 0, 0), (0, 0, 0)))
\end{aligned}$$

and

$$F^{(3)}(x^*)[h]^3 = -162a^3h_1h_2h_3. \tag{2.27}$$

Thus

$$\mathrm{Ker}^3 F^{(3)}(x^*) \quad = \quad lin\left\{\begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix}\right\} \cup lin\left\{\begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix}\right\} \cup lin\left\{\begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}\right\}.$$

As an element h we can take for example one of the vectors:

$$h = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \; h = \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix}, \; h = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}.$$

For instance for $h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ we obtain

$$F^{(3)}(x^*)[h]^2 = 0.$$

If we take $\overline{h} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then we have $F^{(3)}(x^*)[h]\left[\overline{h}\right] \neq 0$ because

$$F^{(3)}(x^*)\left[h, \overline{h}\right] = \left(0, -27a^3, 0\right).$$

But $F^{(3)}(x^*)\left[\overline{h}\right]^2 = 0$, $F^{(3)}(x^*)\left[\overline{h}\right]^2[h] = 0$ and also $F^{(3)}(x^*)[h]^2\left[\overline{h}\right] = 0$. At last we obtain that $h \in T_1 M(x^*)$.

3. Conclusion. The overall aim of this paper consists in the development and application of p-regularity theory in differential geometry problems related to the theory of curves and spaces. The basic features of p-regularity theory were presented in [5, 3, 4]. In the first part of this paper, we investigated the p-regular methods and procedures by using concepts, models and techniques proposed in [1, 2]. In the second part, we developed a modified Brezhneva–Tret'yakov theorem in such situation where a tangent cone is not two-sided. Several theoretical examples were studied to illustrate the fundamental notions of p-regularity.

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