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Boundedness and compactness of weighted composition operators between weighted Bergman spaces

Abstract. We study when a weighted composition operator acting between different weighted Bergman spaces is bounded, resp. compact.

1. Introduction. Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}$ and ψ be an analytic function on $\mathbb{D}$. Such maps induce the weighted composition operator

$$C_{\phi,\psi} : H(\mathbb{D}) \to H(\mathbb{D}), \quad f \mapsto \psi(f \circ \phi),$$

where $H(\mathbb{D})$ denotes the space of all analytic functions endowed with the compact-open topology *co*. The study of (weighted) composition operators acting on various spaces of analytic functions has quite a long and rich history since they appear naturally in a variety of problems, see the excellent monographs [5] and [15]. For a deep insight in the recent research on (weighted) composition operators we refer the reader to the following sample of papers as well as the references therein: [12], [10], [1], [2], [3], [4], [13], [14], [11].

We say that a function $v : \mathbb{D} \to (0, \infty)$ is a *weight* if it is bounded and continuous. For a weight v we consider the space

$$A_{v,2} := \left\{ f \in H(\mathbb{D});\ \|f\|_{v,2} := \left(\int_{\mathbb{D}} |f(z)|^2 v(z)\, dA(z) \right)^{\frac{1}{2}} < \infty \right\},$$

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where $dA(z)$ is the normalized area measure such that area of $\mathbb{D}$ is 1. Endowed with norm $\|\cdot\|_{v,2}$ this is a Banach space. Thus, $A_{1,2}$ denotes the usual Bergman space. An introduction to the concept of Bergman spaces is given in [9] and [7].

In [16] we characterized the boundedness of weighted composition operators acting between weighted Bergman spaces generated by weights given as the absolute value of holomorphic functions using a method by Čučković and Zhao [6]. In this paper we study boundedness and compactness of weighted composition operators acting between different weighted Bergman spaces generated by a quite general class of radial weights.

2. Preliminaries. In this section we collect some geometrical data of the open unit disk as well as some well-known basic facts we will need to treat the problem mentioned above. For $a, z \in \mathbb{D}$ let $\sigma_a(z)$ be the Möbius transformation of $\mathbb{D}$ which interchanges 0 and a, that is

$$\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

Obviously

$$\sigma_a'(z) = -\frac{1 - |a|^2}{(1 - \bar{a}z)^2} \text{ for every } z \in \mathbb{D}.$$

It turned out that the Carleson measure is a very useful tool when studying (weighted) composition operators on weighted Bergman spaces, see [6] and [16]. Recall that a positive Borel measure μ on $\mathbb{D}$ is said to be a *Carleson measure* on the Bergman space if there is a constant $C > 0$ such that, for any $f \in A_{1,2}$

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \le C\|f\|_{1,2}^2.$$

For an arc I in the unit circle $\partial\mathbb{D}$ let $S(I)$ be the Carleson square defined by

$$S(I) = \left\{ z \in \mathbb{D};\ 1 - |I| \le |z| < 1,\ \frac{z}{|z|} \in I \right\}.$$

The following result is well known. In its present form it is taken from [6] (see there Theorem A) and [8].

Theorem 1 ([6] Theorem A). *Let μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent.*

(i) *There is a constant $C_1 > 0$ such that, for any positive subharmonic function f we have that*

$$\int_{\mathbb{D}} f^2(z) \, d\mu(z) \le C_1 \int_{\mathbb{D}} f^2(z) \, dA(z).$$

(ii) *There is a constant $C_2 > 0$ such that, for any arc $I \subset \partial\mathbb{D}$,*

$$\mu(S(I)) \le C_2|I|^2.$$

(iii) *There is a constant $C_3 > 0$ such that, for every $a \in \mathbb{D}$,*

$$\int_{\mathbb{D}} |\sigma_a'(z)|^2 \, d\mu(z) \leq C_3.$$

The study of the compactness of the operator $C_{\phi,\psi}$ requires the following proposition which can be found in the book of Cowen and MacCluer, see [5].

Proposition 2 (Cowen–MacCluer [5], Proposition 3.11). *The operator $C_{\phi,\psi}$: $A_{v,2} \to A_{w,2}$ is compact if and only if for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $A_{v,2}$ such that $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, then $C_{\phi,\psi} f_n \to 0$ in $A_{w,2}$.*

In the sequel we consider the following class of weights. Let ν be a holomorphic function on $\mathbb{D}$, non-vanishing, strictly positive on $[0,1[$ and satisfying $\lim_{r\to 1} \nu(r) = 0$. Then we define the weight v by

$$v(z) := \nu(|z|^2)$$

for every $z \in \mathbb{D}$.

Next, we give some illustrating examples of weights of this type:

(i) Consider $\nu(z) = (1-z)^\alpha$, $\alpha \geq 1$. Then the corresponding weight is the so-called standard weight $v(z) = (1-|z|^2)^\alpha$.

(ii) Select $\nu(z) = e^{-\frac{1}{(1-z)^\alpha}}$, $\alpha \geq 1$. Then we obtain the weight $v(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$.

(iii) Choose $\nu(z) = \sin(1-z)$ and the corresponding weight is given by $v(z) = \sin(1-|z|^2)$.

(iv) Let $\nu(z) = (1-\log(1-z))^q$, $q \leq -1$, for every $z \in \mathbb{D}$. Hence we obtain the weight $v(z) = (1-\log(1-|z|^2))^q$, $q \leq -1$, for every $z \in \mathbb{D}$.

For a fixed point $a \in \mathbb{D}$ we introduce a function $\nu_a(z) := \nu(\overline{a}z)$ for every $z \in \mathbb{D}$. Since ν is holomorphic on $\mathbb{D}$, so is the function ν_a.

It can be easily seen that each weight, which is defined as above, is subharmonic.

3. Boundedness. This section is devoted to the study of the boundedness of $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$. In fact, the following result corresponds to the results obtained in [6] and [16]. Actually, the idea to use Carleson measures is due to [6].

Theorem 3. *Let v be a weight as defined above such that*

$$M := \sup_{a\in\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{v(z)}{|\nu_a(z)|} < \infty.$$

Then the weighted composition operator $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ is bounded if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\sigma_a'(\phi(z))|^2}{|\nu_a(\phi(z))|} w(z) |\psi(z)|^2 \, dA(z) < \infty.$$

Proof. First, we assume that $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ is bounded. Now, fix $a \in \mathbb{D}$ and put $f_a(z) = \frac{-\sigma_a'(z)}{\nu_a(z)^{\frac{1}{2}}}$ for every $z \in \mathbb{D}$. Then

$$\|f_a\|_{v,2}^2 = \int_{\mathbb{D}} \frac{|\sigma_a'(z)|^2}{|\nu_a(z)|} v(z)\, dA(z) \le M$$

for every $a \in \mathbb{D}$ and the constant M is independent of the choice of the point a. The boundedness of the operator $C_{\phi,\psi}$ yields that

$$\|C_{\phi,\psi} f_a\|_{w,2}^2 = \int_{\mathbb{D}} \frac{|\sigma_a'(\phi(z))|^2}{|\nu_a(\phi(z))|} w(z)|\psi(z)|^2\, dA(z) \le C\|f_a\|_{v,2}^2 \le CM$$

for every $a \in \mathbb{D}$. Finally,

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\sigma_a'(\phi(z))|^2}{|\nu_a(\phi(z))|} w(z)|\psi(z)|^2\, dA(z) < \infty,$$

as desired.

Conversely, we assume that

$$K := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\sigma_a'(\phi(z))|^2}{|\nu_a(\phi(z))|} w(z)|\psi(z)|^2\, dA(z) < \infty.$$

Obviously, this yields that $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\sigma_a'(\phi(z))|^2 w(z) \frac{|\psi(z)|^2}{v(\phi(z))}\, dA(z) \le K < \infty$. Putting $d\nu_{v,w,\psi} \circ \phi^{-1}$ and changing variable $s = \phi(z)$, this is equivalent with

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\sigma_a'(s)|^2\, d\mu_{v,w,\psi}(s) < \infty.$$

By Theorem 1 this holds if and only if there is a constant $C > 0$ such that

$$\int_{\mathbb{D}} g^2(s)\, d\mu_{v,w,\psi}(s) \le C \int_{\mathbb{D}} g^2(s)\, dA(s) \tag{3.1}$$

for every positive subharmonic function g. Since

$$\begin{aligned}\int_{\mathbb{D}} g^2(\phi(z))|\psi(z)|^2 \frac{w(z)}{v(\phi(z))}\, dA(z) &= \int_{\mathbb{D}} g^2(\phi(z))\, d\nu_{v,w,\psi}(z)\\ &= \int_{\mathbb{D}} g^2(s)\, d\mu_{v,w,\psi}(s),\end{aligned}$$

(3.1) is equivalent with

$$\int_{\mathbb{D}} \frac{g^2(\phi(z))}{v(\phi(z))} |\psi(z)|^2 w(z)\, dA(z) \le C \int_{\mathbb{D}} g^2(z)\, dA(z).$$

Next, put $f(z) := \frac{g(z)}{v^{\frac{1}{2}}(z)}$ for every $z \in \mathbb{D}$. Now, if $\int_{\mathbb{D}} g^2(z)\, dA(z) \le K_1 < \infty$, then, obviously we can find a constant $L > 0$ such that

$$\int_{\mathbb{D}} v(z) f^2(z)\, dA(z) \le L.$$

Hence

$$\int_{\mathbb{D}} f^2(\phi(z))|\psi(z)|^2 w(z)\, dA(z) \le C \int_{\mathbb{D}} f^2(z) v(z)\, dA(z)$$

for every positive subharmonic function f on $\mathbb{D}$ as defined above. Then obviously

$$\int_{\mathbb{D}} |f(\phi(z))|^2 |\psi(z)|^2 w(z)\, dA(z) \le C \int_{\mathbb{D}} |f(z)|^2 v(z)\, dA(z).$$

for every $f \in A_{v,2}$. □

4. Compactness.

Proposition 4. *Let v be a weight and $K := \sup_{z\in\mathbb{D}} w(z)|\psi(z)|^2 < \infty$. Moreover, let the weighted composition operator $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ be bounded. If for every $K \subset \mathbb{D}$ there is $\varepsilon > 0$ such that $\frac{w(z)}{v(\phi(z))}|\psi(z)|^2 < \varepsilon$ for every $z \in \mathbb{D}\backslash K$, then the operator $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ is compact.*

Proof. The idea is to use Proposition 2. Thus, fix a bounded sequence $(f_n)_n \subset A_{v,2}$ such that $(f_n)_n$ converges to zero uniformly on the compact subsets of $\mathbb{D}$. We have to show that $\|C_{\phi,\psi} f_n\|_{w,2} \to 0$ if $n \to \infty$. However,

$$\begin{aligned}
\|C_{\phi,\psi} f_n\|_{w,2}^2 &= \int_{\mathbb{D}} |f_n(\phi(z))|^2 |\psi(z)|^2 w(z)\, dA(z) \\
&\le \int_{\mathbb{D}_r} |f_n(\phi(z))|^2 w(z)|\psi(z)|^2\, dA(z) \\
&\quad + \int_{\mathbb{D}\backslash\mathbb{D}_r} |f_n(\phi(z))|^2 \frac{w(z)|\psi(z)|^2}{v(\phi(z))} v(\phi(z))\, dA(z) \\
&\le K \sup_{|z|\le r} |f_n(\phi(z))| + \sup_{|z|>r} \frac{w(z)|\psi(z)|^2}{v(\phi(z))} \|f_n\|_{v,2}^2,
\end{aligned}$$

where $\mathbb{D}_r = \{z \in \mathbb{D};\ |z| \le r\}$. Finally, the claim follows. □

Proposition 5. *Let v be a weight as defined above such that*

$$M := \sup_{z\in\mathbb{D}} \sup_{a\in\mathbb{D}} \frac{v(z)}{|\nu_a(z)|} < \infty.$$

If the operator $C_{\phi,\psi} : A_{v,2} \to A_{w,2}$ is compact, then

$$\limsup_{|a|\to 1} \int_{\mathbb{D}} \frac{|\sigma_a'(\phi(z))|^2}{|\nu_a(\phi(z))|} w(z)|\psi(z)|^2\, dA(z) = 0.$$

Proof. Consider the function

$$f_a(z) = \frac{-\sigma_a'(z)}{\nu(\overline{a}z)^{\frac{1}{2}}} \text{ for every } z \in \mathbb{D}.$$

Then $\|f_a\|_{v,2}^2 \leq M$ for every $a \in \mathbb{D}$ and $f_a \to 0$ uniformly on the compact subsets of $\mathbb{D}$. Hence, by Proposition 2

$$\|C_{\phi,\psi} f_a\|_{w,2}^2 = \int_{\mathbb{D}} \frac{|\sigma_a'(\phi(z))|^2}{|\nu_a(\phi(z))|} w(z)|\psi(z)|^2 \, dA(z) \to 0$$

if $|a| \to 1$. Hence the claim follows. □

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