Trace parameters for Teichmüller space of genus 2 surfaces and mapping class group

Abstract. We obtain a representation of the mapping class group of genus 2 surface in terms of a coordinate system of the Teichmüller space defined by trace functions.

1. Introduction. We identify $\text{PSL}(2, \mathbb{R})$ with the group of orientation-preserving isometries of the upper half plane $\mathbb{H} = \{ z \in \mathbb{C} : \Im z > 0 \}$ equipped with the hyperbolic metric $|dz|/(\Im z)$.

A Fuchsian subgroup $G$ of $\text{PSL}(2, \mathbb{R})$ is said to be of type $(2; -;-; -)$ ([5, p. 38]) if $\mathbb{H}/G$ is a closed surface of genus 2 and the projection $\pi : \mathbb{H} \to \mathbb{H}/G$ is an unbranched covering. $G$ has a canonical generator system or a marking $E = (A, B, C, D)$ which satisfies

$$[A, B][C, D] = 1,$$

where $[a, b] = aba^{-1}b^{-1}$ is the commutator of $a$ and $b$, and 1 stands for the unit matrix. We call the pair $(G, E)$ a marked Fuchsian group of type $(2; -;-; -)$. Two marked Fuchsian groups $(G_1, E_1)$ and $(G_2, E_2)$ are equivalent if there exists a matrix $P \in \text{PSL}(2, \mathbb{R})$ such that

$$A_2 = P^{-1}A_1P, \quad B_2 = P^{-1}B_1P, \quad C_2 = P^{-1}C_1P, \quad D_2 = P^{-1}D_1P,$$

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where $E_j = (A_j, B_j, C_j, D_j)$, $j = 1, 2$. The Teichmüller space $\mathcal{T}_2$ of type $(2; -; -; -)$ is the space of all equivalence classes of marked Fuchsian groups of type $(2; -; -; -)$. Each marked Fuchsian group $(G, E)$ can be represented by a tuple $(A, B, C, D)$ of matrices in $SL(2, \mathbb{R})$ such that

$$
(1.1) \quad \text{tr}A > 0, \, \text{tr}B > 0, \, \text{tr}C > 0 \, \text{and} \, \text{tr}D > 0.
$$

Therefore, for the rest of this paper, we always assume that $E = (A, B, C, D)$ consists of matrices satisfying (1.1). In this case $\text{tr}AB$ and $\text{tr}CD$ are both positive (this follows from [5, 33.17 (b)]). In [3] we considered the following traces as functions of $[(G, E) = (A, B, C, D)]$ in $\mathcal{T}_2$:

$$
(1.2) \quad a = \text{tr}A, \, b = \text{tr}B, \, z = \text{tr}AB, \, u = -\text{tr}ACDc^{-1},
$$

$$
\quad v = -\text{tr}ACD^2, \, w = -\text{tr}ACD, \, t = \text{tr}CD.
$$

Since all non trivial elements of $G$ are hyperbolic, their traces take values in $\mathbb{R}_{>2} = \{x : x > 2\}$. It is shown in [3] (see also [4]) that the mapping $\Phi: \mathcal{T}_2 \to \mathbb{R}_{>2}^7$ defined by $\Phi[(G, E)] = (a, b, z, u, v, w, t)$ is an embedding and $a, b, z, u, v, w, t$ satisfy the identity

$$
(1.3) \quad awt + a^2 + w^2 + t^2 + K^2 + S^2 + 4 - w\sqrt{(K^2 + 4)(S^2 + 4)} = 0,
$$

where

$$
K = \sqrt{abz - a^2 - b^2 - z^2} \, \text{and} \, S = \sqrt{uvu - u^2 - v^2 - t^2}.
$$

The mapping class group $\mathcal{M}C_2$ is the group of isotopy classes of orientation-preserving homeomorphisms of the orientable closed surface $S$ of genus 2. It is a subgroup of outer automorphisms of the fundamental group of $S$ (see [5]). $\mathcal{M}C_2$ acts on the Teichmüller space $\mathcal{T}_2$ by changing the marking. The purpose of this paper is to describe a generating system of $\mathcal{M}C_2$ by using the coordinate-system $(a, b, z, u, v, w, t)$. It is an interesting observation that $\mathcal{M}C_2$ acts on $\mathcal{T}_2$ as a group of rational transformations.

2. Trace identities.

2.1. Basic trace identities. The matrices $A, B$ and $C$ in $SL(2, \mathbb{R})$ satisfy the following identities (see [2, §3.4]):

$$
(11) \quad \text{tr}A = \text{tr}A^{-1},
$$

$$
(12) \quad \text{tr}AB + \text{tr}AB^{-1} = \text{tr}AtrB,
$$

$$
(13) \quad \text{tr}ABC = \text{tr}AtrBC + \text{tr}BtrCA + \text{tr}CtrAB - \text{tr}AtrBtrC - \text{tr}ACB.
$$

We shall use repeatedly the following identities, which are consequences of (11), (12) and (13) above:

$$
(2.1a) \quad \text{tr}[A, B] = \text{tr}ABA^{-1}B^{-1} = (\text{tr}A)^2 + (\text{tr}B)^2 + (\text{tr}AB)^2 - \text{tr}AtrBtrAB - 2,
$$

$$
(2.1b) \quad \text{tr}ABCB = \text{tr}ABtrBC + \text{tr}AC - \text{tr}AtrC,
$$

$$
(2.1c) \quad \text{tr}ABCB^{-1} = \text{tr}AtrC - \text{tr}AC - \text{tr}ABtrBC + \text{tr}BtrABC.
$$
Let $G$ be a group generated by a finite number of matrices $A_1, \ldots, A_n \in SL(2, \mathbb{R})$ and
\[(2.2)\] 
\[S = \{\text{tr}(A_{i_1}A_{i_2} \cdots A_{i_r}) : 1 \leq i_1 < i_2 < \cdots < i_r \leq n, 1 \leq r \leq n\}.\]

Then the following fact is well known (see [2, §3.5]).

**Lemma 2.1.** Let $g \in G$. Then $\text{tr} g$ is an integer polynomial in $S$.

### 2.2. Trace identities for genus 2 surface.

Let $E = (A, B, C, D)$ be a marking of a Fuchsian group $G$ of type $(2; -; -; -)$. Let $c = x_1 = \text{tr}C$ and $d = x_2 = \text{tr}D$, $x_3 = \text{tr}AC$, $x_4 = \text{tr}AD$, $x_5 = \text{tr}BC$, $x_6 = \text{tr}BD$, $x_7 = \text{tr}ABC$, $x_8 = \text{tr}ABD$, $x_9 = \text{tr}BCD$ and $x_{10} = \text{tr}ABCD$. Then the set $S$ for $G$ with respect to $(A, B, C, D)$ is
\[S = \{a, b, c, d, z, x_3, x_4, x_5, x_6, t, x_7, x_8, x_9, x_{10}\}.\]

The purpose of this section is to find expressions of $x_1, \ldots, x_{10}$ in $\{a, b, z, u, v, w, t\}$ of (1.2). Then by Lemma 2.1 we can express the trace of any element of $G$ in $\{a, b, z, u, v, w, t\}$. We shall apply this fact to obtain a representation of the mapping class group $\mathcal{MC}_2$ via rational transformations.

1. Since $[A, B] = [C, D]^{-1}$, we obtain by (2.1a)
\[(2.3)\]
\[abz - a^2 - b^2 - z^2 = cdt - c^2 - d^2 - t^2.\]

Note that $\text{tr}[A, B] = a^2 + b^2 + z^2 - abz - 2 < -2$, since $G$ is discrete (see, for example [5, 33 D]). In what follows $K = \sqrt{abz - a^2 - b^2 - z^2}$.

2. From $BAB^{-1} = CDC^{-1}D^{-1}A$ and the basic identity (I3) we obtain
\[a = \text{tr}((ACD) \cdot C^{-1} \cdot D^{-1}) = -wt + cx_3 - ud + wcd - a.\]

and hence
\[(2.4)\]
\[2a + wt - cx_3 + ud - wcd = 0.\]

3. From (I2), $v = -\text{tr}ACD \cdot D = -(\text{tr}ACD\text{tr}D - \text{tr}AC) = wd + x_3$ and so
\[(2.5)\]
\[x_3 = v - dw.\]

From this and (2.4) it follows that
\[(2.6)\]
\[2a + wt - cv + ud = 0.\]

4. From (I3),
\[-u = \text{tr}A \cdot CD \cdot C^{-1} = ad + t(\text{tr}AC^{-1}) - wc - atc - x_4\]
\[= ad + t(ac - x_3) - wc - atc - x_4.\]

It follows from this and (2.5) that
\[(2.7)\]
\[x_4 = u + ad - tx_3 - wc = u + ad - tw + twd - cw.\]

By substituting $d = u^{-1}(cv - 2a - wt)$ (see (2.6)) into (2.3) we obtain
\[(uvt - u^2 - v^2)c^2 - (2a + wt)(tu - 2v)c - (K^2 + t^2)u^2 - (2a + tw)^2 = 0.\]
Hence
\[ \text{tr}(2.11) \]
(5)
From (I2) and (2.1c) applied to
\[(2.9)\]
Now from (2.8) we obtain
\[ c = \frac{(2a + tw)(at - 2c) + u\sqrt{(2a + tw)^2(t^2 - 4) + 4(K^2 + t^2)(S^2 + t^2)}}{2(S^2 + t^2)}, \]
\[ d = \frac{cv - 2a - wt}{u} \]
where \( S = \sqrt{wut - u^2 - v^2 - t^2} \). By using (1.3) we see that \((2a + tw)^2(t^2 - 4) + 4(K^2 + t^2)(S^2 + t^2)\) equals
\[ \left((t^2 - 4)w + 2\sqrt{(S^2 + 4)(K^2 + 4)}\right)^2 = \left((t^2 - 4)w + \frac{2(aw + a^2 + t^2 + K^2 + S^2 + 4)}{w}\right)^2. \]
Now from (2.8) we obtain
\[ c = \frac{(K^2 + S^2 + t^2 + a^2 + 4)u + w(2atu - 2av - uw + t^2uw - twv)}{w(S^2 + t^2)}, \]
\[ d = \frac{(K^2 + S^2 + t^2 + a^2 + 4)v + w(2au + twu - vw)}{w(S^2 + t^2)}. \]
By (2.5), (2.7) and (2.9), we can obtain the expressions of \( x_3 = \text{tr}AC \) and \( x_4 = \text{tr}AD \) in \((a, b, z, u, v, w, t)\),
\[ x_3 = -\frac{uw(2a + tw) + v(4 + a^2 + K^2 - w^2)}{S^2 + t^2} \]
\[ x_4 = \frac{(ad + u - cw) + t(4 + a^2 + K^2 - w^2)v + wu(2a + tw)}{S^2 + t^2}. \]
(5) From (I2) and (2.1c) applied to \( BCDC^{-1} \) we obtain
\[ \text{tr}B^{-1}(CD)^{-1} = bd - \text{tr}BCDC^{-1} \]
\[ = bd - (bd - x_6 + x_5t + cx_9) = x_6 + tx_5 - cx_9. \]
From (I3), \( \text{tr}B^{-1}CD = bt - x_9 \). Then, from the trace of \( AB^{-1}A^{-1} = B^{-1}CD \cdot C^{-1} \cdot D^{-1} \), (I2), (I3) and (2.11),
\[ b = (\text{tr}B^{-1}CD)t + \text{ctr}B^{-1}C + dtr(B^{-1}CD \cdot C^{-1}) - (\text{tr}B^{-1}CD)cd - b \]
\[ = (bt - x_9)(t - cd) + c(bc - x_5) + d(x_6 + tx_5 - cx_9) - b. \]
Hence
\[ (dt - c)x_5 + dx_6 - tx_9 = 2b - bt^2 + bcdt - bc^2. \]
(6) From (I2), \( trA^{-1}CD = at + w \), and from (I2) and (I3),
\[
trB^{-1}A^{-1} \cdot C \cdot D = zt + ctrABD^{-1} + dtrABC^{-1} - zcd - trB^{-1}A^{-1}DC
\]
\[
= zt + c(zd - x_h) + d(zc - x_7) - zcd - trB^{-1}A^{-1}DC
\]
\[
= zt + cdz - dx_7 - cx_8 - trB^{-1}A^{-1}DC.
\]
Substituting these into the next equation obtained from \( B^{-1}A^{-1}DC = A^{-1} \cdot B^{-1} \cdot CD \) and (I3),
\[
trB^{-1}A^{-1}DC = a(trB^{-1}CD + btrA^{-1}CD + zt - abt - trB^{-1}A^{-1}CD
\]
\[
= a(bt - x_9) + b(at + w) + zt - abt
\]
\[
- zt - cdz + dx_7 + cx_8 + trB^{-1}A^{-1}DC,
\]
we obtain
\[
dx_7 + cx_8 - ax_9 = -abt - bw + cdz.
\]
(7) From \( B^{-1}CDC^{-1} = trAB^{-1}A^{-1}D, trB^{-1}(CDC^{-1}) \) equals
\[
trAB^{-1}A^{-1}D = trBtrAA^{-1}D - trABA^{-1}D = bd - trDABA^{-1}
\]
\[
= bd - (trBtrD - trBD - trBAtRD + trAtRBD)
\]
\[
= x_6 + zx_4 - ax_8.
\]
Here we have used (I2) and (2.1c). Then from (2.11),
\[
tx_5 + ax_8 - cx_9 = zx_4.
\]
(8) From \( BA^{-1}B^{-1}C = A^{-1}DCD^{-1} \) and (I2), we have
\[
ac - trBAB^{-1}C = trBA^{-1}B^{-1}C = trA^{-1}DCD^{-1} = ac - trADC^{-1},
\]
and hence \( trCBAB^{-1} = trADC^{-1} \). We have by using (2.1c)
\[
trCBAB^{-1} = trCtrA - trAC - trBCtrAB + trBtrCBA
\]
\[
= ac - x_3 - zx_5 + b(trCtriBA + trBtrCA + trAtRCB
\]
\[
- trAtRtrC - trABC)
\]
\[
= ac - x_3 - zx_5 + bcz + b^2x_3 + abx_5 - ab^2c - bx_7
\]
and
\[
trADC^{-1} = trAtRtrC - trAC - trADtrDC + trDtrADC
\]
\[
= ac - x_3 - tx_4 + d(trAtRCD + trDtrAC + trCtriAD
\]
\[
- trAtRtrC - trACD)
\]
\[
= ac - x_3 - tx_4 + adt + d^2x_3 + cdx_4 - ad^2c + wd.
\]
Thus we obtain
\[
(z - ab)x_5 + bx_7 = (b^2 - d^2)x_3 + (t - cd)x_4 + bcz - ab^2c - adt + ad^2c - wd.
\]
We use $C^{-1}BA = \text{tr} DC^{-1}D^{-1}AB$. Then from (I2) and (I3),

$$\text{tr} C^{-1}BA = zc - \text{tr} CBA = zc - (cz + bx_3 + ax_5 - abc - x_7) = bx_3 - ax_5 + abc + x_7.$$ 

From (I2) and (2.1c) this equals

$$\text{tr}(DC^{-1}D^{-1})AB = cz - \text{tr} ABDCD^{-1} = cz - (\text{tr} AB\text{tr}C - \text{tr} ABC - \text{tr} A\text{tr}D\text{tr}C + \text{tr} D\text{tr}(AB \cdot D \cdot C)) = x_7 + tx_8 - d(zt + dx_7 + cx_8 - zcd - x_{10}).$$

Hence we obtain

$$-ax_5 + d^2x_7 + (cd - t)x_8 - dx_{10} = -abc + bx_3 - dtz + cd^2z.$$ 

(10) We use $D^{-1}C^{-1}B = C^{-1}D^{-1}ABA^{-1}$. From (I2), $\text{tr} D^{-1}C^{-1}B = bt - x_9$ and from (I2), (2.1c) and (I3),

$$\text{tr} C^{-1}D^{-1}ABA^{-1} = tb - \text{tr}(DC)ABA^{-1} = (dx_5 + cx_6 + bt - bcd - x_9) + z(dx_3 + cx_4 + at - acd + w) - a(zt + dx_7 + cx_8 - zcd - x_{10})$$

we obtain

$$dx_5 + cx_6 - adx_7 - acx_8 + ax_{10} = bcd - zdx_3 - zcx_4 - zw.$$ 

Let

$$M = \begin{pmatrix}
dt - c & d & 0 & 0 & -t & 0 \\
0 & 0 & d & c & -a & 0 \\
t & 0 & 0 & a & -c & 0 \\
z - ab & 0 & b & 0 & 0 & 0 \\
-a & 0 & d^2 & cd - t & 0 & -d \\
d & c & -ad & -ac & 0 & a
\end{pmatrix}, \quad \tilde{x} = \begin{pmatrix}
x_5 \\
x_6 \\
x_7 \\
x_8 \\
x_9 \\
x_{10}
\end{pmatrix}.$$ 

and

$$\bar{v} = \begin{pmatrix}
2b - bt^2 + bcdt - bc^2 \\
- abst - bw + cdz \\
(b^2 - d^2)x_3 + (t - cd)x_4 + bcz - ab^2c - adt + acd^2 - wd \\
-abc + bx_3 - dzt + cd^2z \\
bc^2 - dxx_3 - cxx_4 - zw
\end{pmatrix}.$$
From the results (5)–(10) we obtain $M\vec{x} = \vec{v}$. The matrix $M$ is singular, if $a = c$. However, by using (2.4) and (2.7) we can deduce:

\[
\begin{align*}
    x_5 &= \frac{c(2b + a^2 b - 2az + bK^2) - tus + dv(ab + z + zK^2) - v(ab + zK^2)}{K^2 + a^2}, \\
    x_6 &= \frac{2(adz - bd) - u(ab + K^2z) + tv(ab + z + K^2z) + (c - dt)w(ab + z + K^2z)}{K^2 + a^2}, \\
    x_7 &= \frac{-2cz - btu + avz + wd(b - az)}{K^2 + a^2}, \\
    x_8 &= \frac{d(K^2 + a^2 + 2) + auz + vt(b - az) + w(bc - bdt - acz + adtz)}{K^2 + a^2}, \\
    x_9 &= \frac{t(2b + a^2 b - 2az + bK^2) + dvz + w(ab + zK^2) + u(cz - dtz)}{K^2 + a^2}, \\
    x_{10} &= \frac{-2t^2 + b(c - dt)u + bdv - awz}{K^2 + a^2}.
\end{align*}
\]

Expressions for $x_3$ and $x_4$ are obtained in (2.10).

### 3. Mapping class group.

Let $G$ be a group of type $(2; -; -; -)$ and $E = (A, B, C, D)$ a marking (or a canonical generator system) of $G$. We consider the following changes of marking:

\[
\begin{align*}
    \omega_1(E) &= (AB^{-1}, B, C, D), & \omega_2(E) &= (B, BA, C, D), \\
    \omega_3(E) &= (B^{-1}CA, B, C, B^{-1}CD), & \omega_4(E) &= (A, B, CD^{-1}, D), \\
    \omega_5(E) &= (A, B, C, DC).
\end{align*}
\]

Each $\omega_j$ induces an automorphism of $G$, which is also denoted by $\omega_j$. The table below shows the images of the elements in the leftmost column under $\omega_j$.

<table>
<thead>
<tr>
<th></th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
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<tbody>
<tr>
<td>A</td>
<td>$AB^{-1}$</td>
<td>A</td>
<td>$B^{-1}CA$</td>
<td>A</td>
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<tr>
<td>B</td>
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<tr>
<td>AB</td>
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<td>$B^{-1}CAB$</td>
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<td>$ACDC^{-1}$</td>
<td>$AB^{-1}CD^{-1}$</td>
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<td>$B^{-1}CAB^{-1}CD^{-1}$</td>
<td>$ACDC^{-1}$</td>
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<td>$AB^{-1}CD^2$</td>
<td>$ACD^2$</td>
<td>$B^{-1}CA(B^{-1}CD)^{2}$</td>
<td>$ACD$</td>
<td>$AC(DC)^{2}$</td>
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<tr>
<td>$ACD$</td>
<td>$AB^{-1}CD$</td>
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<td>$B^{-1}CAB^{-1}CD$</td>
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<td>CD</td>
<td>CD</td>
<td>CD</td>
<td>$CB^{-1}CD$</td>
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</table>
Let $\omega_{j*} \in MC_2$ denote the mapping class induced by $\omega_j$. Then $\omega_{1*}, \ldots, \omega_{5*}$ generate $MC_2$ and satisfy the following relations [1, Theorem 4.8]:

$$\omega_{i*} \omega_{j*} = \omega_{j*} \omega_{i*}$$

if $|i - j| \geq 2$, $1 \leq i, j \leq 5$,

$$\omega_{j*} \omega_{j*+1*} \omega_{j*} = \omega_{j*+1*} \omega_{j*} \omega_{j*+1*}$$

($j = 1, 2, 3, 4$),

$$(\omega_{1*} \omega_{2*} \omega_{3*} \omega_{4*} \omega_{5*})^6 = 1,$$

$$\omega_{1*} \omega_{2*} \omega_{3*} \omega_{4*} \omega_{5*} = 1.$$ 

In this section we represent the action of $\omega_{j*}$ on $T_2$ in the variables $a, b, z, u, v, w, t$. More precisely, when $(A_j, B_j, C_j, D_j) = \omega_j(A, B, C, D)$, we express

$$a_j = trA_j, \quad b_j = trB_j, \quad z_j = trA_j B_j, \quad u_j = -trA_j C_j D_j C_j^{-1},$$

$$v_j = -trA_j C_j D_j^2, \quad w_j = -trA_j C_j D_j, \quad t_j = trC_j D_j$$

by using $a, b, z, u, v, w, t$. However, for the case of $\omega_3$ we modify the signs of some traces to obtain positive values.

(Case of $\omega_{1*}$) By using basic trace identities we have $trAB^{-1} = trAtrB - trAB = ab - z$,

$$w_1 = -trAB^{-1} CD = -trBtrACD + trABCD = bw + x_{10},$$

$$u_1 = -trAB^{-1} CDC^{-1} = -trBtrACDC^{-1} + tr(AB)CDC^{-1} (\because (I2))$$

$$= bu + (trABtrD - trABD - trABCtrCD + trCtrABCD) (\because (2.1c))$$

$$= bu + zd - x_8 - tx_7 + cx_{10},$$

and

$$v_1 = -trAB^{-1} CD^2 = -trBtrACD^2 + trABCD^2 (\because (I2))$$

$$= bv + (trABCDtrD - trABC) (\because (I2))$$

$$= bv + dx_{10} - x_7.$$ 

Hence

$$\omega_{1*}(a, b, z, u, v, w, t) = (ab - z, b, a, u_1, v_1, w_1, t).$$

(Case of $\omega_{2*}$) Since $trABA = trABtrA - trB = za - b$,

$$\omega_{2*}(a, b, z, u, v, w, t) = (a, z, az - b, u, v, w, t).$$

(Case of $\omega_{3*}$) First we remark that $trB^{-1} CA < 0$ and $trB^{-1} CD < 0$. To see $trB^{-1} CA < 0$, for example, note that $(AB^{-1}, B)$ is a marking for a group of type $(1; 0; 0; 1)$ and $trA$ and $trB$ are positive. Then we have $trAB^{-1} > 0$. Then $(AB^{-1}, C)$ is a marking for a group of type $(0; 0; 0; 3)$. Since $trAB^{-1}$ and $trC$ are positive, $trAB^{-1} C < 0$ (see [5, Section 33 A and D]). The calculation for $\omega_{3*}$ is the most complicated: By using the basic trace identities we have

$$a_3 = trB^{-1} CA = trBtrAC - trABC = bx_3 - x_7.$$
\[ w_3 = -\text{tr}(B^{-1}C)(AC)(B^{-1}C)D \]
\[ = -\text{tr}(AC)(B^{-1}C)D(B^{-1}C) \]
\[ = -\text{tr}ACB^{-1}C\text{tr}B^{-1}CD - \text{tr}ACD + \text{tr}AC\text{tr}D \]
\[ = -(\text{tr}B\text{tr}AC^2 - \text{tr}ACBC)(\text{tr}B\text{tr}CD - \text{tr}BCD) + w + dx_3 \]
\[ = -[b(cx_3 - a) - (x_3x_5 + z - ab)][(bt - x_9) + w + dx_3 \]
\[ = (x_3x_5 + z - bcx_3)(bt - x_9) + w + dx_3, \]

\[ u_3 = -\text{tr}(B^{-1}C)(AC)(B^{-1}C)(DC^{-1}) \]
\[ = -\text{tr}(AC)(B^{-1}C)(DC^{-1})(B^{-1}C) \]
\[ = -\text{tr}ACB^{-1}C\text{tr}B^{-1}CDC^{-1} - \text{tr}ACD^{-1} + \text{tr}AC\text{tr}DC^{-1} \]
\[ = -(\text{tr}AC\text{tr}B^{-1}C - \text{tr}AB)(\text{tr}B\text{tr}D - \text{tr}BCD^{-1}) + u + x_3(cd - t) \]
\[ = -(x_3(bc - x_5) - z)[bd - (bd - x_6 - tx_5 + cx_9)] + u + x_3(cd - t) \]
\[ = (x_3x_5 + z - bcx_3)(x_6 + tx_5 - cx_9) + u + x_3(cd - t), \]

\[ v_3 = -\text{tr}B^{-1}CAC(B^{-1}C)^2 \]
\[ = -\text{tr}B^{-1}CD\text{tr}B^{-1}CACB^{-1}CD + \text{tr}B^{-1}CAC \]
\[ = (bt - x_9)[(x_3x_5 + z - bcx_3)(bt - x_9) + w + dx_3] + (bc - x_5)x_3 - z, \]

\[ t_3 = \text{tr}CB^{-1}CD = \text{tr}CB^{-1}\text{tr}CD - \text{tr}BD = (bc - x_3)t - x_6. \]

In this case \( a_3, x_3, v_3 \) and \( t_3 \) are negative. We modify the sign of these parameters and obtain

\[ \omega_{3+}(a, b, z, u, v, w, t) = (-a_3, b, -x_3, u_3, -v_3, w_3, -t_3). \]

(Case of \( \omega_{4+} \)) For the expression of \( \omega_{4+} \) we have easily

\[ \omega_{4+}(a, b, z, u, v, w, t) = (a, b, z, u, w, -x_3, c). \]

(Case of \( \omega_{5+} \)) Since \( -\text{tr}ACDC = -\text{tr}C\text{tr}ACD + \text{tr}ACDC^{-1} = cw - u, \)

\[ v_5 = -\text{tr}AC(DC)^2 = -\text{tr}CD\text{tr}ACDC + \text{tr}AC \]
\[ = -t(\text{tr}C\text{tr}ACD - \text{tr}ACDC^{-1}) + x_3 \]
\[ = cw - tu + x_3, \]

and \( \text{tr}CDC = ct - d, \) we have

\[ \omega_{5+}(a, b, z, u, v, w, t) = (a, b, z, w, cw - tu + x_3, cw - u, ct - d). \]

Now we conclude
Theorem 3.1. The mapping classes \( \omega_1 \ast, \omega_2 \ast, \omega_3 \ast, \omega_4 \ast, \omega_5 \ast \) are represented by the following rational maps in variables \( a, b, z, u, v, w, t \):

\[
\begin{align*}
\omega_1 \ast (a, b, z, u, v, w, t) &= (ab - z, b, a, u, v, w, t) \\
\omega_2 \ast (a, b, z, u, v, w, t) &= (a, az - b, u, v, w, t) \\
\omega_3 \ast (a, b, z, u, v, w, t) &= (-bx_3 + x_7, b, -x_3, u_3, -v_3, w_3, -btc + x_5t + x_6) \\
\omega_4 \ast (a, b, z, u, v, w, t) &= (a, b, z, u, w, -x_3, c) \\
\omega_5 \ast (a, b, z, u, v, w, t) &= (a, b, z, w, cw - tu + x_3, cw - u, ct - d),
\end{align*}
\]

where \( c, d, x_3, x_4, x_5, x_6 \) and \( x_7 \) are given in (2.9) and (2.10) and (2.12).

As it is shown in Section 2, \( x_1 = c, x_2 = d, ..., x_{10} \) are all rational functions in \( (a, b, z, u, v, w, t) \). Hence the inverse mappings of \( \omega_j \ast (j = 1, ..., 5) \) are also rational mappings. The expressions in (3.2) in \( (a, b, z, u, v, t) \), especially the one for \( \omega_3 \ast \), are very complicated.

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