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Properties of functions concerned with Carathéodory functions

ABSTRACT. Let \mathcal{P}_n denote the class of analytic functions $p(z)$ of the form $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots$ in the open unit disc \mathbb{U} . Applying the result by S. S. Miller and P. T. Mocanu (J. Math. Anal. Appl. **65** (1978), 289–305), some interesting properties for $p(z)$ concerned with Carathéodory functions are discussed. Further, some corollaries of the results concerned with the result due to M. Obradović and S. Owa (Math. Nachr. **140** (1989), 97–102) are shown.

1. Introduction. Let \mathcal{A}_n denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n = 1, 2, 3, \dots)$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. If a function $f(z) \in \mathcal{A}_n$ satisfies

$$(1.2) \quad \operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}),$$

then $f(z)$ is said to be starlike with respect to the origin in \mathbb{U} . We denote by \mathcal{S}_n^* the subclass of \mathcal{A}_n consisting of functions $f(z)$ which are starlike with respect to the origin in \mathbb{U} . From the definition of the class \mathcal{S}_n^* , we see that

2000 *Mathematics Subject Classification.* Primary 30C45.

Key words and phrases. Analytic, starlike, Carathéodory function.

if $f(z) \in \mathcal{A}_n$ satisfies

$$(1.3) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_n^*$. We denote by \mathcal{T}_n^* the subclass of \mathcal{S}_n^* consisting of $f(z)$ satisfying (1.3).

Obradović and Owa [5] have shown the following result:

Theorem A. *If $f(z) \in \mathcal{A}_1$ satisfies $f(z)f'(z) \neq 0$ for $0 < |z| < 1$ and*

$$(1.4) \quad \left| 1 + \frac{zf''(z)}{f'(z)} \right| < \frac{5}{4} \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{T}_1^*$.

In order to discuss our results, we have to recall here the following lemma due to Miller and Mocanu [3] (also due to Jack [2]):

Lemma 1.1. *Let*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a_n \neq 0)$$

be analytic in \mathbb{U} . If there exists a point $z_0 \in \mathbb{U}$ on the circle $|z| = r < 1$ such that

$$(1.5) \quad \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)|,$$

then we can write

$$(1.6) \quad z_0 w'(z_0) = m w(z_0),$$

where m is real and $m \geq n$.

Example 1.1. We consider the function $w(z)$ given by

$$(1.7) \quad w(z) = z^n + \frac{e^{i\theta}}{n+1} z^{n+1} \quad (n = 1, 2, 3, \dots).$$

Then, it follows that

$$(1.8) \quad \max_{|z| \leq |z_0|} |w(z)| = \max_{|z| \leq |z_0|} |z|^n \left| 1 + \frac{e^{i\theta} z}{n+1} \right| \leq r^n \left(1 + \frac{r}{n+1} \right)$$

for $z_0 = r e^{-i\theta} \in \mathbb{U}$. This shows that $|w(z)|$ attains its maximum value at a point $z_0 \in \mathbb{U}$ on the circle $|z| = r$. For such a point $z_0 = r e^{-i\theta}$, we have that

$$(1.9) \quad \frac{z_0 w'(z_0)}{w(z_0)} = \frac{z_0^n (n + e^{i\theta} z_0)}{z_0^n \left(1 + \frac{e^{i\theta} z_0}{n+1} \right)} = \frac{(n+1)(n+r)}{n+1+r} = m \geq n.$$

Let \mathcal{P}_n be the class of functions $p(z)$ of the form

$$(1.10) \quad p(z) = 1 + \sum_{k=n}^{\infty} c_k z^k \quad (c_n \neq 0)$$

which are analytic in \mathbb{U} . We also denote by \mathcal{Q}_n the subclass of \mathcal{P}_n consisting of $f(z)$ which satisfy

$$(1.11) \quad |p(z) - 1| < 1 \quad (z \in \mathbb{U}).$$

Since $p(z) \in \mathcal{Q}_n$ shows that $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$), $p(z) \in \mathcal{Q}_n$ is said to be a Carathéodory function in \mathbb{U} (see Carathéodory [1]).

2. Conditions for the classes \mathcal{Q}_n and \mathcal{T}_n^* . Applying Lemma 1.1, we discuss some conditions for $p(z) \in \mathcal{P}_n$ to be in the class \mathcal{Q}_n .

Theorem 2.1. *If $p(z) \in \mathcal{P}_n$ satisfies*

$$(2.1) \quad \operatorname{Re} \left(p(z) + \alpha \frac{zp'(z)}{p(z)} \right) < \sqrt{\alpha n} |p(z)| \quad (z \in \mathbb{U})$$

for some real $\alpha > 0$, then $p(z) \in \mathcal{Q}_n$.

Proof. Note that $p(z) \neq 0$ ($z \in \mathbb{U}$) with the condition (2.1). Let us define the function $w(z)$ by

$$(2.2) \quad p(z) = 1 + w(z) \quad (z \in \mathbb{U})$$

for $p(z) \in \mathcal{P}_n$. Then $w(z)$ is analytic in \mathbb{U} and

$$(2.3) \quad w(z) = c_n z^n + c_{n+1} z^{n+1} + \dots$$

It follows that

$$(2.4) \quad p(z) + \alpha \frac{zp'(z)}{p(z)} = 1 + w(z) + \frac{\alpha zw'(z)}{1 + w(z)}$$

and that

$$(2.5) \quad \begin{aligned} \frac{1}{|p(z)|} \operatorname{Re} \left(p(z) + \alpha \frac{zp'(z)}{p(z)} \right) \\ = \frac{1}{|1 + w(z)|} \operatorname{Re} \left(1 + w(z) + \frac{\alpha zw'(z)}{1 + w(z)} \right) < \sqrt{\alpha n} \end{aligned}$$

for $z \in \mathbb{U}$.

We suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$(2.6) \quad \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, Lemma 1.1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = m e^{i\theta}$ ($m \geq n$). For such a point z_0 , we have that

$$\begin{aligned}
 \frac{1}{|p(z_0)|} \operatorname{Re} \left(p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) &= \frac{1}{|1 + e^{i\theta}|} \operatorname{Re} \left(1 + e^{i\theta} + \frac{\alpha m e^{i\theta}}{1 + e^{i\theta}} \right) \\
 (2.7) \qquad \qquad \qquad &= \frac{1}{\sqrt{2(1 + \cos \theta)}} \left(1 + \cos \theta + \frac{\alpha m}{2} \right) \\
 &= \frac{1}{\sqrt{2}} \left(\sqrt{1 + \cos \theta} + \frac{\alpha m}{2\sqrt{1 + \cos \theta}} \right) \\
 &\geq \sqrt{\alpha m} \geq \sqrt{\alpha n}.
 \end{aligned}$$

This contradicts the condition (2.1). Therefore, there is no such point $z_0 \in \mathbb{U}$. This means that $p(z) \in \mathcal{Q}_n$. \square

Corollary 2.1. *If $f(z) \in \mathcal{A}_n$ satisfies $f(z)f'(z) \neq 0$ for $0 < |z| < 1$ and*

$$(2.8) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{z f'(z)}{f(z)} + \alpha \left(1 + \frac{z f''(z)}{f'(z)} \right) \right\} < \sqrt{\alpha n} \left| \frac{z f'(z)}{f(z)} \right| \quad (z \in \mathbb{U})$$

for some real $\alpha > 0$, then $f(z) \in \mathcal{T}_n^*$.

Proof. Letting $p(z) = \frac{z f'(z)}{f(z)}$ in Theorem 2.1, we have that

$$p(z) + \alpha \frac{z p'(z)}{p(z)} = (1 - \alpha) \frac{z f'(z)}{f(z)} + \alpha \left(1 + \frac{z f''(z)}{f'(z)} \right).$$

The proof of the corollary follows from the above. \square

Next we derive

Theorem 2.2. *If $p(z) \in \mathcal{P}_n$ satisfies $\operatorname{Re} p(z) \neq 0$ ($z \in \mathbb{U}$) and*

$$(2.9) \quad \operatorname{Re} \left(p(z) + \alpha \frac{z p'(z)}{p(z)} \right) < \left(1 + \frac{\alpha n}{4} \right) \operatorname{Re} p(z) \quad (z \in \mathbb{U})$$

for some real $\alpha > 0$, then $p(z) \in \mathcal{Q}_n$.

Proof. Define the function $w(z)$ by (2.2) for $p(z) \in \mathcal{P}_n$. Then, $w(z)$ is analytic in \mathbb{U} ,

$$w(z) = c_n z^n + c_{n+1} z^{n+1} + \dots,$$

and

$$(2.10) \quad \frac{\operatorname{Re} \left(p(z) + \alpha \frac{z p'(z)}{p(z)} \right)}{\operatorname{Re} p(z)} = \frac{\operatorname{Re} \left(1 + w(z) + \frac{\alpha z w'(z)}{1 + w(z)} \right)}{\operatorname{Re}(1 + w(z))} < 1 + \frac{\alpha n}{4}$$

($z \in \mathbb{U}$). If we suppose that there exists a point $z_0 \in \mathbb{U}$ on the circle $|z| = r < 1$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

we can write that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = m e^{i\theta}$. This shows that

$$(2.11) \quad \frac{\operatorname{Re} \left(p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right)}{\operatorname{Re} p(z_0)} = \frac{1 + \cos \theta + \frac{\alpha m}{2}}{1 + \cos \theta} \geq 1 + \frac{\alpha m}{4} \geq 1 + \frac{\alpha n}{4}.$$

Since (2.11) contradicts our condition (2.9), $|w(z)| < 1$ for all $z \in \mathbb{U}$. This means that $p(z) \in \mathcal{Q}_n$. \square

If we take $p(z) = \frac{z f'(z)}{f(z)}$ in Theorem 2.2, we have

Corollary 2.2. *If $f(z) \in \mathcal{A}_n$ satisfies $\operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) \neq 0$ ($z \in \mathbb{U}$) and*

$$(2.12) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{z f'(z)}{f(z)} + \alpha \left(1 + \frac{z f''(z)}{f'(z)} \right) \right\} < \left(1 + \frac{\alpha n}{4} \right) \operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right)$$

($z \in \mathbb{U}$) for some real $\alpha > 0$, then $f(z) \in \mathcal{T}_n^*$.

Corollary 2.3. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$(2.13) \quad \operatorname{Re} \left(\frac{z f''(z)}{f'(z)} \right) < \operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) + \frac{n-2}{n} \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{T}_n^*$.

Proof. If we write

$$\frac{z f'(z)}{f(z)} = 1 + w(z) \quad (f(z) \in \mathcal{A}_n),$$

we see that $w(z)$ is analytic in \mathbb{U} and

$$w(z) = c_n z^n + c_{n+1} z^{n+1} + \dots$$

For such a function $w(z)$, we see that

$$(2.14) \quad \operatorname{Re} \left(\frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right) = \operatorname{Re} \left(\frac{z w'(z)}{1 + w(z)} - 1 \right) < \frac{n-2}{2} \quad (z \in \mathbb{U}).$$

Supposing that there exists a point $z_0 \in \mathbb{U}$ on the circle $|z| = r < 1$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

we can write that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = m e^{i\theta}$. Therefore, we have

$$(2.15) \quad \operatorname{Re} \left(\frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 f'(z_0)}{f(z_0)} \right) = \operatorname{Re} \left(\frac{k e^{i\theta}}{1 + e^{i\theta}} - 1 \right) = \frac{k}{2} - 1 \geq \frac{n-2}{2},$$

which contradicts the condition (2.13). This implies that $f(z) \in \mathcal{T}_n^*$. \square

Example 2.1. Let us consider the function $p(z)$ given by

$$(2.16) \quad p(z) = 1 + a_n z^n \quad (z \in \mathbb{U})$$

for some $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, where a_n satisfies

$$a_n^3 + 2a_n - 1 \leq 0 \quad (0 < a_n < 1).$$

Then $p(z) \in \mathcal{P}_n$ and $p(z) \neq 0$ ($z \in \mathbb{U}$). It is clear that $p(z)$ satisfies the condition (2.9) in Theorem 2.2 for $z = 0$.

Let us put $z = e^{i\theta}$ for $p(z)$. Then we see that

$$(2.17) \quad \operatorname{Re} \left(p(z) + \alpha \frac{z p'(z)}{p(z)} \right) = 1 + a_n \cos n\theta + \frac{\alpha n a_n (a_n + \cos n\theta)}{a_n^2 + 1 + 2a_n \cos n\theta}$$

and

$$(2.18) \quad \left(1 + \frac{\alpha n}{4} \right) \operatorname{Re} p(z) = \left(1 + \frac{\alpha n}{4} \right) (1 + a_n \cos n\theta).$$

This gives us that

$$(2.19) \quad \begin{aligned} & \left(1 + \frac{\alpha n}{4} \right) \operatorname{Re} p(z) - \operatorname{Re} \left(p(z) + \alpha \frac{z p'(z)}{p(z)} \right) \\ &= \frac{\alpha n (1 + 2a_n \cos n\theta + a_n^3 \cos n\theta + 2a_n^2 \cos^2 n\theta)}{4(a_n^2 + 1 + 2a_n \cos n\theta)} \\ &\geq \frac{\alpha n (1 - 2a_n - a_n^3)}{4(a_n^2 + 1 + 2a_n \cos n\theta)} \geq 0. \end{aligned}$$

Therefore, the function $p(z)$ satisfies the condition (2.9) for all $z \in \mathbb{U}$. Indeed, we see that

$$|p(z) - 1| = |a_n z^n| < a_n < 1 \quad (z \in \mathbb{U}).$$

Furthermore, if we define the function $f(z) \in \mathcal{A}_n$ by

$$(2.20) \quad \frac{z f'(z)}{f(z)} = 1 + a_n z^n$$

with some real a_n ($0 < a_n < 1$) satisfying

$$a_n^3 + 2a_n - 1 \leq 0,$$

then we have that

$$(2.21) \quad f(z) = z e^{\frac{\alpha n}{n} z^n}$$

which satisfies the condition (2.12) in Corollary 2.2.

If we consider the function

$$g(x) = x^3 + 2x - 1 \quad (0 < x < 1),$$

we see that $g(0) = -1 < 0$ and $g\left(\frac{1}{2}\right) = \frac{1}{8} > 0$. Therefore, there exists some real x ($0 < x < 1$) such that $g(x) \leq 0$. Indeed, we see that

$$0.4533 < x < 0.4534.$$

3. Properties for the classes \mathcal{P}_n and \mathcal{A}_n . We discuss some properties for functions in the classes \mathcal{P}_n and \mathcal{A}_n .

Theorem 3.1. *If $p(z) \in \mathcal{P}_n$ satisfies*

$$(3.1) \quad \int_{|z|=r} \left| \operatorname{Re} \left(\frac{zp'(z)}{p(z)} \right) \right| d\theta < \pi$$

for $z = re^{i\theta}$ ($0 < r < 1$), then $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$).

Proof. It follows from (3.1) that

$$(3.2) \quad \int_{|z|=r} \left| \operatorname{Re} \left(\frac{zp'(z)}{p(z)} \right) \right| d\theta = \int_0^{2\pi} \left| \frac{d \arg p(z)}{d\theta} \right| d\theta = \int_{|z|=r} |d \arg p(z)| < \pi.$$

This implies that $\operatorname{Re} p(z) > 0$ for $|z| = r < 1$. Applying the maximum principle for harmonic functions, we obtain that $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$). \square

From Theorem 3.1, we have

Corollary 3.1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$(3.3) \quad \int_{|z|=r} \left| \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right| d\theta < \pi$$

for $z = re^{i\theta}$ ($0 < r < 1$), then $f(z) \in \mathcal{S}_n^*$.

Further, applying the same method as the proof by Umezawa [5] and Nunokawa [3], we derive the following result:

Theorem 3.2. *If $f(z) \in \mathcal{A}_1$ satisfies*

$$(3.4) \quad -\frac{\beta}{4\beta - 1} < \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) < \beta \quad (z \in \mathbb{U})$$

for some real $\beta \geq \frac{1}{4}$, then $\operatorname{Re} f'(z) > 0$ ($z \in \mathbb{U}$).

Proof. We note that if $f'(z_0) = 0$ for some $z_0 \in \mathbb{U}$, then $f(z)$ does not satisfy the condition (3.4). This shows that $f'(z) \neq 0$ for all $z \in \mathbb{U}$. Applying the same method by Umezawa [5] and Nunokawa [3], we have that

$$(3.5) \quad \int_{|z|=r} \frac{zf''(z)}{f'(z)} d\theta = \int_{|z|=r} \frac{zf''(z)}{f'(z)} \frac{dz}{iz} = -i \int_{|z|=r} \frac{zf''(z)}{f'(z)} dz = 0.$$

We denote by \mathcal{C}_1 the part of the circle $|z| = r$ on which

$$(3.6) \quad \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \geq 0$$

and

$$(3.7) \quad \int_{\mathcal{C}_1} d \arg z = x.$$

On the other hand, let us denote by \mathcal{C}_2 the part of the circle $|z| = r$ on which

$$(3.8) \quad \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) < 0$$

and

$$(3.9) \quad \int_{\mathcal{C}_2} d \arg z = 2\pi - x.$$

Putting

$$(3.10) \quad y_1 = \int_{\mathcal{C}_1} \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) d\theta = \int_{\mathcal{C}_1} \left(\frac{d \arg f'(z)}{d\theta} \right) d\theta$$

and

$$(3.11) \quad -y_2 = \int_{\mathcal{C}_2} \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) d\theta = \int_{\mathcal{C}_2} \left(\frac{d \arg f'(z)}{d\theta} \right) d\theta,$$

we have that $y_1 - y_2 = 0$.

In view of the condition (3.4), we obtain that

$$y_1 < \beta x \quad \text{and} \quad y_2 < \frac{\beta}{4\beta - 1}(2\pi - x).$$

If $y_1 \geq \frac{\pi}{2}$, then $y_2 = y_1 \geq \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta x$. On the other hand, we have that

$$(3.12) \quad y_2 < \frac{\beta}{4\beta - 1}(2\pi - x) < \frac{2\pi\beta - \frac{\pi}{2}}{4\beta - 1} = \frac{\pi}{2}.$$

This contradicts the inequality $y_2 \geq \frac{\pi}{2}$. Therefore, $y_1 = y_2 < \frac{\pi}{2}$. Consequently, we obtain that

$$(3.13) \quad y_1 + y_2 = \int_{|z|=r} \left| \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \right| d\theta = \int_{|z|=r} |d \arg f'(z)| < \pi,$$

which implies that $\operatorname{Re} f'(z) > 0$ ($z \in \mathbb{U}$). \square

Finally, letting $\beta \rightarrow \infty$, $\beta = \frac{1}{4}$ and $\beta = \frac{1}{2}$ in Theorem 3.2, we have the following corollary.

Corollary 3.2. *If $f(z) \in \mathcal{A}_1$ satisfies one of the following conditions*

$$(3.14) \quad \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) > -\frac{1}{4} \quad (z \in \mathbb{U}),$$

$$(3.15) \quad \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) < \frac{1}{4} \quad (z \in \mathbb{U}),$$

$$(3.16) \quad \left| \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) \right| < 1 \quad (z \in \mathbb{U}),$$

then $\operatorname{Re} f'(z) > 0$ ($z \in \mathbb{U}$).

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Received April 11, 2012