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Remarks on some recent results about polynomials with restricted zeros

Abstract. We point out certain flaws in two papers published in Ann. Univ. Mariae Curie-Skłodowska Sect. A, one in 2009 and the other in 2011. We discuss in detail the validity of the results in the two papers in question.

1. Introduction. The following result was proved by Govil [3].

Theorem A. *Let $P(z)$ be a polynomial of degree n having all its zeros in the disk $|z| \le k$ for some $k \ge 1$. Then*

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^n} \max_{|z|=1} |P(z)| . \tag{1}$$

The result is best possible and equality holds for $P(z) = z^n + k^n$.

The next result is also due to Govil [4, p. 184, Theorem D].

Theorem B. *Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n having all its zeros on $|z| = k$ for some $k \le 1$. Then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |p(z)| . \tag{2}$$

In [2] the authors state and I quote: "In this paper, we consider a class of polynomials $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu \le n$ and generalize as well as

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improve upon Theorem A and also generalize Theorem B by proving the following results". They state their "so-called generalizations" of Theorem A, etc. as follows.

Theorem 1. *If* $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ *is a polynomial of degree* n, *having all its zeros in the disk* $|z| \le k$, $k \ge 1$, *then*

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^{n-\mu+1}} \max_{|z|=1} |P(z)| \,. \tag{3}$$

The result is best possible and equality holds for

$$P(z) = (z^{n-\mu+1} + k^{n-\mu+1})^{\frac{n}{n-\mu+1}} \,.$$

Theorem 2. *If* $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \le \mu < n$ *is a polynomial of degree* n, *having all its zeros on* $|z| = k$, $k \le 1$, *then*

$$\max_{|z|=1}|P'(z)| \le \frac{n}{k^{n-\mu+1}} \left(\frac{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1+k^{\mu-1}) + n|c_n|k^{\mu-1}(1+k^{\mu+1})} \right) \max_{|z|=1} |P(z)| \,.$$

Theorem 3. *If* $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ *is a polynomial of degree* n, *having all its zeros in the disk* $|z| \le k$, $k \ge 1$, *then*

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{1+k^{n-\mu+1}} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \right\} . \tag{4}$$

The result is best possible and equality holds for

$$P(z) = (z^{n-\mu+1} + k^{n-\mu+1})^{\frac{n}{n-\mu+1}} \,.$$

2. Some comments on Theorems 1, 2 and 3. Unfortunately, Theorems 1 and 3 are false. As regards Theorem 2, its proof is based on a lemma that is erroneous.

To see that Theorem 1 is false, let us consider the example $P(z) := z^n + k^n$. This is a polynomial which does have the form $c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ with

$$c_0 = k^n,\ c_\nu = 0 \ \text{ for } \ \nu = \mu, \ldots, n-1\,, \ \text{ and } \ c_n = 1\,,$$

where μ can be taken to be any integer in $\{1, 2, \ldots, n-1\}$. Besides, it has all its zeros on $|z| = k$. Clearly, $\max_{|z|=1} |P(z)| = 1 + k^n$ and $\max_{|z|=1} |P'(z)| = n$. Thus, if (3) was true, then we would have

$$n \ge \frac{n}{1+k^{n-\mu+1}}(1+k^n)$$

for any $\mu \in \{1, \ldots, n-1\}$, which amounts to saying that $k^{n-\mu+1} \ge k^n$ for any $\mu \in \{1, \ldots, n-1\}$. For $k > 1$, this is obviously false except when $\mu = 1$. Even if $k^{n-\mu+1} = k^n$ when $\mu = 1$ or $k = 1$, it is of no significance since when $\mu = 1$ or $k = 1$, the so-called Theorem 1 says nothing more than what Theorem A does.

In the face of this counter-example, the authors of [2] might claim that in $z^n + k^n$, which is our counter-example, the coefficients $c_1, \ldots, c_{n-1}$ are all zero whereas in Theorem 1, c_μ is supposed to be different from 0. So, we shall give a counter-example in which $c_\mu \neq 0$.

Take any $a > 1$ and consider the polynomial $P(z) := z^n + \delta z^\mu + a^n$, where δ is supposed to be positive and small. Since the zeros of P are continuous functions [5, p. 9] of δ and those of $z^n + a^n$ all lie on $|z| = a$ the polynomial P has all its zeros in $|z| \leq k$, where $|k - a| \to 0$ as $\delta \to 0$. Now, note that

$$\max_{|z|=1} |P(z)| = 1 + \delta + a^n \quad \text{and} \quad \max_{|z|=1} |P'(z)| = n + \delta\mu\,.$$

Then, according to Theorem 1, we would have

$$\left(\mu - \frac{n}{1 + k^{n-\mu+1}}\right)\delta \geq \left(\frac{1 + a^n}{1 + k^{n-\mu+1}} - 1\right)n = \left(\frac{a^n - k^{n-\mu+1}}{1 + k^{n-\mu+1}}\right)n\,. \tag{5}$$

As $\delta \to 0$,

$$\left(\frac{a^n - k^{n-\mu+1}}{1 + k^{n-\mu+1}}\right)n \to \left(\frac{a^n - a^{n-\mu+1}}{1 + a^{n-\mu+1}}\right)n\,,$$

which is strictly positive if $1 < \mu < n - 1$. Hence, for any such μ, there exists a positive number δ_0 such that

$$\left(\frac{a^n - k^{n-\mu+1}}{1 + k^{n-\mu+1}}\right)n > \frac{1}{2}\left(\frac{a^n - a^{n-\mu+1}}{1 + a^{n-\mu+1}}\right)n \quad \text{for} \quad 0 < \delta < \delta_0\,.$$

Now, from (5) it follows that

$$\left(\mu - \frac{n}{1 + k^{n-\mu+1}}\right)\delta > \frac{1}{2}\left(\frac{a^n - a^{n-\mu+1}}{1 + a^{n-\mu+1}}\right)n$$

for $0 < \delta < \delta_0$. This cannot be true since the expression on the left-hand side of the inequality tends to 0 as $\delta \to 0$ whereas the expression on the right-hand side is a positive constant. The second sentence in the statement of Theorem 1 is: "The result is best possible and equality holds for $P(z) = (z^{n-\mu+1} + k^{n-\mu+1})^{\frac{n}{n-\mu+1}}$". This statement implicitly presumes that $(z^{n-\mu+1} + k^{n-\mu+1})^{n/(n-\mu+1)}$ is a polynomial. However, for $(z^{n-\mu+1} + k^{n-\mu+1})^{n/(n-\mu+1)}$ to be a polynomial, n must be divisible by $n - \mu + 1$. Surprisingly, the authors do not seem to realize this. This remark also applies to the second sentence in the statement of Theorem 3.

Since Theorem 1 is false, as we have shown above, Theorem 3 cannot be true either because it clearly says more than what Theorem 1 does.

The above comments clearly debunk Theorems 1 and 3 of Dewan and Hans.

2.1. The principal error in the proofs of Theorems 1 and 3. Since Theorems 1 and 3 are invalid, there must be something wrong with their proofs. This had to be looked into, which we did. We found a serious mistake in the proof of Lemma 1 of their paper [2]. It is applied to obtain Lemma 2, which the authors use to prove Theorem 1. Here is what Lemma 1 of Dewan and Hans says.

Lemma 1. *If* $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ *is a polynomial of degree* n*, having all its zeros in the disk* $|z| \le k$, $k \ge 1$*, then for* $|z| = 1$

$$k^{n+\mu-3}|Q'(z)| \le |P'(k^2 z)|\,, \tag{6}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

The polynomial $P(z) := z^n + k^n$ satisfies the conditions of Lemma 1 with any μ such that $1 \le \mu < n$. For this polynomial, (6) reduces to $k^{\mu-1} \le 1$, which clearly does not hold for any $\mu > 1$ if $k > 1$. This shows that Lemma 1 is false for $2 \le \mu \le n-1$ and $k > 1$.

The authors use the faulty Lemma 1 to prove Lemma 2, stated as follows.

Lemma 2. *If* $P(z) = c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ *is a polynomial of degree* n*, having all its zeros in the disk* $|z| \le k$, $k \ge 1$*, then*

$$\max_{|z|=1} |Q'(z)| \le k^{n-\mu+1} \max_{|z|=1} |P'(z)|\,,$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

The example $z^n + k^n$ shows that this lemma is also false for $2 \le \mu \le n-1$ and $k > 1$.

We note that the proof of Theorem 1, as given by Dewan and Hans, uses Lemma 2. Since Lemma 2 is deduced from Lemma 1, it is desirable to identify the error in the proof of Lemma 1 as presented by the authors on pages 57–58 of [2]. So, we shall do that.

Using a standard argument, the authors conclude that ([2, p. 58], see (2.3))

$$k^{n-1}|Q'(z/k)| \le k|P'(kz)| \quad \text{for} \quad |z| \ge 1\,.$$

This is fine. Since $c_1 = \cdots = c_{\mu-1} = 0$, this can be written as

$$k^{n-1}|Q'(z/k)| \le k\left|(kz)^{\mu-1}\sum_{\nu=\mu}^{n} \nu c_\nu (kz)^{\nu-\mu}\right| \quad \text{for} \quad |z| \ge 1\,.$$

In particular, the authors say (see inequality (2.4) of their paper) that

$$k^{n-1}|Q'(z/k)| \le k^\mu \left|\sum_{\nu=\mu}^{n} \nu c_\nu (kz)^{\nu-\mu}\right| \tag{7}$$

for $|z| = 1$. We agree with this. Next, they say that $\sum_{\nu=\mu}^{n} \nu c_\nu (kz)^{\nu-\mu} \ne 0$ in $|z| > 1$ and we agree once again. Then they make the bizarre assertion that by maximum modulus principle it (by which they mean (7)) also holds for $|z| > 1$. They overlook that for this to be true

$$\frac{k^{n-1}|Q'(z/k)|}{k^{\mu}\left|\sum_{\nu=\mu}^{n} \nu c_{\nu}(kz)^{\nu-\mu}\right|}$$

must tend to a finite limit as $z \to \infty$. Except in the case where $c_0 = 0$ the above mentioned quotient tends to infinity as $z \to \infty$. Thus, *the proof of Lemma 1 is based on a false application of the maximum modulus principle*.

We are sorry to add that the authors apply Lemma 2 to prove another lemma which they state as follows.

Lemma 3. *If* $P(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ *is a polynomial of degree* n*, having no zeros in the disk* $|z| \leq k$, $k \leq 1$*, then*

$$k^{n-\mu+1} \max_{|z|=1} |P'(z)| \leq \max_{|z|=1} |Q'(z)|\,,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Once again, the example $z^n + k^n$ shows that this lemma is also invalid for $2 \leq \mu \leq n-1$ and $k < 1$.

3. Another related paper. The authors have gone on to use their faulty Lemmas 1, 2 and 3 in another paper, namely [1] published in Ann. Univ. Mariae Curie-Skłodowska Sect. A in the year 2011. As we shall explain, Theorems 1 and 2 of [1] are not true. The results in [1] involve the notion of polar derivative. The polar derivative of a polynomial $P(z)$ with respect to a point α, denoted by $D_\alpha P(z)$, is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)\,.$$

Theorem 1 of [1] can be stated as follows. Because of its obvious relationship with Theorem 2 of [2], stated above as Theorem 2, we shall name it Theorem 2b.

Theorem 2b. *If* $P(z) = c_n z^n + \sum_{j=\mu}^{n} c_{n-j} z^{n-j}$, $1 \leq \mu < n$, *is a polynomial of degree* n *having all its zeros on* $|z| = k$, $k \leq 1$*, then for every real or complex number* α *with* $|\alpha| \geq k$*, we have*

$$\begin{aligned} &\max_{|z|=1} |D_\alpha P(z)| \\ &\leq \frac{n(|\alpha| + k^{\mu})}{k^{n-\mu+1}} \frac{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}{\mu|c_{n-\mu}|(1 + k^{\mu-1}) + n|c_n|k^{\mu-1}(1 + k^{\mu+1})} \max_{|z|=1} |P(z)|\,. \end{aligned} \tag{8}$$

For $P(z) := z^n + k^n$, $k \leq 1$, which is a polynomial satisfying the conditions of Theorem 2b, inequality (8) says that

$$k^n + |\alpha| \leq \frac{k^{\mu} + |\alpha|}{k^{n-\mu+1}} \frac{k^{2\mu}}{k^{\mu-1}(1 + k^{\mu+1})} \tag{9}$$

and so *a fortiori* $k^{n-2\mu}(k^n + |\alpha|) \leq k^\mu + |\alpha|$, that is

$$k^{2n-2\mu} + k^{n-2\mu}|\alpha| \leq k^\mu + |\alpha|\,.$$

If $k < 1$, then $k^{2n-2\mu} > k^\mu$ and $k^{n-2\mu} > 1$ for any $\mu > 2n/3$. Thus, (9) and so also (8) cannot hold for any $\mu > 2n/3$. As indicated by the authors (see [1, pp. 6–7, §3]) the proof of Theorem 2b uses Lemma 2 of [1], which is the same as Lemma 3 of [2], cited above. Since Lemma 3 of [2] is false, as we have already indicated, their proof of Theorem 2b is invalid and there is really no need to look for counter-examples to (8) for $\mu \leq 2n/3$.

4. The polynomials considered by Dewan and Hans. It seems that Dewan and Hans overlooked the fact that in inequality (1) of Govil, equality holds for $P(z) := z^n + k^n$, which is a polynomial of the form $P(z) := c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$. To think that they could improve upon (1), by considering polynomials which are of the form $P(z) := c_0 + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, was not a promising idea to start with. They could obtain a stronger conclusion than that of Theorem A only if they considered a class of polynomials which did not contain the polynomial $z^n + k^n$. In fact, there is no *raison d' ètre* for Theorems 1 and 3. Not only are their proofs not correct, their statements are false. The problem with Theorem 2 is of a different nature; its proof uses Lemma 3, which is faulty.

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