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On the birational gonalities of smooth curves

Abstract. Let C be a smooth curve of genus g. For each positive integer r the birational r-gonality $s_r(C)$ of C is the minimal integer t such that there is $L \in \mathrm{Pic}^t(C)$ with $h^0(C,L) = r+1$. Fix an integer $r \geq 3$. In this paper we prove the existence of an integer g_r such that for every integer $g \geq g_r$ there is a smooth curve C of genus g with $s_{r+1}(C)/(r+1) > s_r(C)/r$, i.e. in the sequence of all birational gonalities of C at least one of the slope inequalities fails.

1. Introduction. Let C be a smooth curve of genus g. For each positive integer r the birational r-gonality $s_r(C)$ of C is the minimal integer t such that there is $L \in \mathrm{Pic}^t(C)$ with $h^0(C,L) = r+1$ ([1], §2). In this paper we prove the following result.

Theorem 1. *Fix an integer $r \geq 3$. Then there exists an integer g_r such that for every integer $g \geq g_r$ there is a smooth curve C of genus g with $s_{r+1}(C)/(r+1) > s_r(C)/r$.*

Theorem 1 means that for the curve C at least one slope inequality fails. For any integer $r \geq 1$ the r-gonality of C is the minimal degree of a line bundle L on C with $h^0(C,L) \geq r+1$. Obviously $s_r(C) \geq d_r(C)$ if $r \geq 2$. Equality holds if $d_r(C) < r \cdot d_1(C)$ and C has no non-trivial morphism onto a smooth curve of positive genus. In [6] H. Lange and G. Martens studied

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the slope inequality for the usual gonality sequence of smooth curves (it may fail for some C, but not for a general C).

We work over an algebraically closed base field with characteristic zero.

2. Working inside a Hirzebruch surface. Fix $e \in \mathbb{N}$. Let $F_e \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ denote the Hirzebruch surface ([4], Chapter V, §2). We call $\pi : F_e \to \mathbb{P}^1$ a ruling of F_e. We have $\mathrm{Pic}(F_e) \cong \mathbb{Z}^2$ and take as a basis of $\mathrm{Pic}(F_e)$ a fiber f of π and a section h of π with $h^2 = -e$ (π and h are unique if $e > 0$). For any finite set $S \subset F_e$ let $2S$ denote the first infinitesimal neighborhood of S in F_e, i.e. the closed subscheme of F_e with $(\mathcal{I}_S)^2$ has its ideal sheaf. We have $(2S)_{red} = S$ and $\deg(2S) = 3 \cdot \sharp(S)$. Fix an integer $a \geq 0$. The line bundle $\mathcal{O}_{F_e}(ah+bf)$ is spanned (resp. very ample) if and only if $b \geq ea$ (resp. $b > ea$ and $a > 0$) ([4], V.2.18). We have $h^1(F_e, \mathcal{O}_{F_e}(ah+bf)) = 0$ if and only if $b \geq -1$. If $b \geq ae$, then

$$h^0(F_e, \mathcal{O}_{F_e}(ah+bf)) = (a+1)(2b-ea+2)/2$$

([5], Proposition 2.3). Assume $a > 0$ and $b \geq ae$; if $e = 0$, then assume $b > 0$. Fix any $Y \in |\mathcal{O}_{F_e}(ah+eaf)|$. Since $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h+(-e-2)f)$, the adjunction formula gives

$$\omega_Y \cong \mathcal{O}_Y((a-2)h+(ea-e-2)f).$$

Hence $p_a(Y) = 1 + a(ea-e-2)/2$. We have

$$h^0(F_e, \mathcal{O}_{F_e}(ah+eaf)) = (ea+2)(a+1)/2.$$

To prove Theorem 1 for the integer r we will use as C the normalization of a nodal curve $Y \in |\mathcal{O}_{F_e}(ah+eaf)|$, where $e := r-1$.

Notation 1. For all integers $a \geq 1$ and $e \geq 1$ set $g_{a,e} := 1 + a(ae-2-e)/2$.

Notice that if $a \geq 2$, then $g_{a,e} - g_{a-1,e} = ae - e - 1$.

Lemma 1. *Assume $e \geq 2$. Fix integers a, x. If $x = 0$, assume $a \geq 1$. If $x > 0$, assume $a \geq 5$ and $3x \leq (ea-2e+1)(a-1)/2$. Fix a general $S \subset F_e$ such that $\sharp(S) = x$. Then*

$$h^1(F_e, \mathcal{I}_{2S}(ah+eaf)) = 0,\ h^0(F_e, \mathcal{I}_{2S}(ah+eaf)) = (ea+2)(a+1)/2 - 3x,$$

a general $Y \in |\mathcal{I}_{2S}(ah+eaf)|$ is integral, nodal and with $\mathrm{Sing}(Y) = S$.

Proof. First assume $x = 0$. Since $\mathcal{O}_{F_e}(ah+aef)$ is spanned, Bertini's theorem gives that a general $Y \in |\mathcal{O}_{F_e}(ah+aef)|$ is smooth. Since

$$h^0(F_e, \mathcal{O}_{F_e}(h+ef)) + h^0(F_e, \mathcal{O}_{F_e}((c-1)h+(c-1)rf)) < h^0(F_e, \mathcal{O}_{F_e}(ch+cf))$$

for every integer $c \in \{1, \ldots, a-1\}$ and $|\mathcal{O}_{F_e}(uh+vf)|$ has h in the base locus if $u > 0$ and $v < eu$, Y is also irreducible.

Now assume $x > 0$. Fix a general $S \subset F_e$ such that $\sharp(S) = x$. Since

$$3x \leq h^0(F_e, \mathcal{O}_{F_e}((a-2)h+e(a-2)f)),$$

$e \geq 2$ and $a-2 \geq 3$, a theorem of A. Laface gives

$$h^1(F_e, \mathcal{I}_{2S}((a-2)h + e(a-2)f)) = 0$$

([5], Proposition 5.2 and case $m = 2$ of Theorem 7.2). Hence

$$h^1(F_e, \mathcal{I}_{2S}((a-i)h + e(a-i)f)) = 0$$

for $i = 0, 1$. Hence

$$h^0(F_e, \mathcal{I}_{2S}(ah + eaf)) = (ea+2)(a+1)/2 - 3x.$$

Fix $P \in F_e \setminus S$ and a general $A \in |\mathcal{O}_{F_e}(h+e)f)|$ containing P. The curve A is smooth if $P \notin h$, while $A = h \cup F$ with $F \in |\mathcal{O}_{F_e}(f)|$ if $P \in h$. In all cases we see that $\mathcal{O}_A(ah + eaf)$ is spanned at P (in the case $P \in h$ use the following facts: $\mathcal{O}_h(ah + eah) \cong \mathcal{O}_h$, $F \cong \mathbb{P}^1$, and $\mathcal{O}_{\mathbb{P}^1}(a)$ is spanned). Since $h^1(F_e, \mathcal{O}_{F_e}((a-1)h + e(a-1)f)) = 0$, $P \in A$ and $\mathcal{O}_A(ah + eaf)$ is spanned at P, the exact sequence

$$\begin{aligned} 0 &\to \mathcal{I}_{2S}((a-1)h + e(a-1)f) \to \mathcal{I}_{2S}((a-1)h + e(a-1)f) \\ &\to \mathcal{O}_A(ah + eaf) \to 0 \end{aligned} \tag{1}$$

gives that $\mathcal{I}_{2S}(ah + eaf)$ is spanned at P. Since this is true for all $P \notin S$, Bertini's theorem gives $\mathrm{Sing}(Y) = S$. In particular Y has no multiple component. Fix $P \in S$. Since S is general, we have $P \notin h$. Since $|\mathcal{O}_{F_e}(h + ef)|$ induces a morphism with injective differential at P, $|\mathcal{O}_{F_e}(2h + 2af)|$ spans the jets at P of $\mathcal{O}_{F_e}$ up to order 2. Hence we may find $Y' \in |\mathcal{O}_{F_e}(2h + 2ef)|$ with an ordinary node at P. Since

$$h^1(F_e, \mathcal{I}_{2S}((a-2)h + e(a-2)f)) = 0,$$

we have

$$h^1(F_e, \mathcal{I}_{\{P\} \cup 2(S \setminus \{P\})}((a-2)h + e(a-2)f)) = 0.$$

Hence

$$\begin{aligned} h^0(F_e, &\mathcal{I}_{\{P\} \cup 2(S \setminus \{P\})}((a-2)h + e(a-2)f)) \\ &= h^0(F_e, \mathcal{I}_{2(S \setminus \{P\})}((a-2)h + e(a-2)f)) - 1. \end{aligned}$$

Hence there is $Y'' \in |\mathcal{I}_{2(S \setminus \{P\})}((a-2)h + e(a-2)f)|$ such that $P \notin Y''$. Hence $Y'' \cup Y'$ has an ordinary node at P. Since $Y'' \cup Y' \in |\mathcal{I}_{2S}(ah + eaf)|$, S is finite and Y is general, Y is nodal. Recall that $\mathrm{Sing}(Y) = S$ and that S is general. Since S is general, no pair of points of S is on the same fiber of the ruling of F_e. Hence no fiber of F_e may be an irreducible component of Y. Since $\mathcal{O}_{F_e}(ch + ecf) \cdot \mathcal{O}_{F_e}((a-c)h + e(a-c)f) = ec(a-c)$, we immediately see that Y is irreducible. □

Lemma 2. *Assume $e \geq 2$. Fix integers a, x. If $x = 0$, assume $a \geq 1$. If $x > 0$, assume $a \geq 5$ and $3x \leq (ea - 2e + 1)(a-1)/2$. Fix a general $S \subset F_e$ such that $\sharp(S) = x$ and a general $Y \in |\mathcal{I}_{2S}(ah + eaf)|$. Let $u : C \to Y$ denote the normalization map. The line bundle $u^*(\mathcal{O}_Y(f))$ is spanned* *and* $h^0(C, u^*(\mathcal{O}_Y(f))) = 2$. *Let* $\rho : C \to \mathbb{P}^1$ *be the morphism induced by* $|u^*(\mathcal{O}_Y(f))|$. *Then* ρ *is not composed with an involution, i.e. there are no* (C', ρ', ρ'') *with* C' *a smooth curve,* $\rho' : C \to C'$, $\rho'' : C' \to \mathbb{P}^1$, $\rho = \rho'' \circ \rho'$, $\deg(\rho') \geq 2$ *and* $\deg(\rho'') \geq 2$.

Proof. Obviously $u^*(\mathcal{O}_Y(f))$ is spanned. Since $ae+1-e-2 \geq e(a-2)-1$, Serre's duality gives

$$h^1(F_e, \mathcal{O}_{F_e}(-ah-(ae+1)f)) = h^1(F_e, \mathcal{O}_{F_e}((a-2)h+(ae+1-e-2)f)) = 0.$$

Hence $h^0(Y, \mathcal{O}_Y(f)) = 2$. Since $h^i(F_e, \mathcal{O}_{F_e}) = 0$, $i = 1, 2$, $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h + (-e-2)f))$, Y is nodal and $S = \mathrm{Sing}(Y)$, we have

$$H^0(Y, \omega_Y) \cong H^0(F_e, \mathcal{O}_{F_e}((a-2)h + (ae-e-2)f))$$

and $H^0(C, \omega_C)$ is induced (after deleting the base points) from

$$H^0(F_e, \mathcal{I}_S((a-2)h + (ae-2-e)f)).$$

Hence $h^0(C, u^*(\mathcal{O}_Y(f))) = 2 = h^0(Y, \mathcal{O}_Y(f))$ if and only if

$$h^1(C, u^*(\mathcal{O}_Y(f))) = x + h^1(Y, \mathcal{O}_Y(f)),$$

i.e. if and only if $h^1(F_e, \mathcal{I}_S((a-2)h + (ae-e-3)f)) = 0$. The last equality is true, because S is general and $x \leq (a-1)(ea-2-2e)/2 = h^0(F_e, \mathcal{I}_S((a-2)h + (ae-e-3)f))$.

For any $P \in F_e$ let F_P be the fiber of the ruling of F_e containing P. We fix $P \in F_e \setminus h$ such that $F_P \cap S = \emptyset$. Let $Z \subset F_P$ be the degree two effective divisor with P as its support. Take any $S_1 \subset F_P \setminus \{P, h \cap F_P\}$ such that $\sharp(S_1) = a-2$ and set $Z' := Z \cup S_1$. Taking the inclusion $F_P \hookrightarrow F_e$, we may also see Z' as a degree a zero-dimensional subscheme of F_e.

Claim. $h^1(F_e, \mathcal{I}_{2S \cup Z'}(ah + aef)) = 0$.

Proof of the Claim. Set $T := h \cup F_P \in |\mathcal{O}_{F_e}(h+f)|$. Since $S \cap h = \emptyset$ and $S \cap F_P = \emptyset$, we have $S \cap T = \emptyset$. Hence $(2S \cup Z') \cap T = Z'$. We proved during the proof of Lemma 1 that $h^1(F_e, \mathcal{I}_{2S}((a-1)h + (a-1)ef))) = 0$. Hence $h^1(F_e, \mathcal{I}_{2S}((a-1)h + (ae-e+e-1)f)) = 0$. Notice that

$$\mathcal{I}_{2S}((a-1)h + (ae-e+e-1)f) \cong \mathcal{I}_{2S}(ah + aef)(-T).$$

Since $h^1(F_e, \mathcal{I}_{2S}(ah + aef)) = 0$ (Lemma 1), the Claim is true if

$$h^1(T, \mathcal{I}_{Z',T}(ah + aef)) = 0.$$

The nodal curve T has two irreducible components, h and F_P, and both components are isomorphic to $\mathbb{P}^1$. Since $Z' \cap h = \emptyset$, we have $Z' \cap h \cap F_P = \emptyset$ and hence the $\mathcal{O}_T$-sheaf $\mathcal{I}_{Z'}(ah+aef)$ is a line bundle. Since $Z' \cap h = \emptyset$ and $\mathcal{O}_h(ah+aef) \cong \mathcal{O}_h$, we have $\mathcal{I}_{Z',T}(ah+aef)|h \cong \mathcal{O}_h$. Since $\deg(Z') = a$, we have $\mathcal{I}_{Z',T}(ah + aef) \cap F_P \cong \mathcal{O}_{F_P}$. Hence a Mayer–Vietoris exact sequence gives $h^1(T, \mathcal{I}_{Z',T}(ah + aef)) = 0$, concluding the proof of the Claim.

The Claim is equivalent to

$$h^0(F_e, \mathcal{I}_{2S \cup Z'}(ah + aef)) = h^0(F_e, \mathcal{I}_{2S}(ah + aef)) - a.$$

Set $\Gamma := \bigcup_{Q\in S} F_Q$. We take all $Y \in |\mathcal{I}_{2S}(ah + eaf)|$ containing some Z'. The set of all $P \in F_e$ has dimension 2. For fixed P the set of all $S_1 \subset F_P \setminus F_P \cap (\{P\} \cup h)$ with $\sharp(S_1) = a - 2$ has dimension $a - 2$. Each Y may contain only finitely many schemes Z', because each non-constant morphism $C \to \mathbb{P}^1$ has only finitely many ramification points. Varying first $P \in F_e \setminus (h \cup \Gamma)$ and then all $S_1 \subset F_P \setminus (h \cap F_P \cup \{P\})$ with $\sharp(S_1) = a - 2$, we get that a general $Y \in |\mathcal{I}_{2S}(ah + aef)|$ contains some Z' for some $P \in F_e \setminus (h \cup \Gamma)$. Let $u : C \to \mathbb{P}^1$ be the normalization of any such Y, say containing $Z' = Z \cup S_1$ with $Z \subset F_P$. We saw that $h^0(C, u^*(\mathcal{O}_Y(f))) = 2$. Let $\rho : C \to \mathbb{P}^1$ be the morphism associated to $|u^*(\mathcal{O}_Y(f))|$. Notice that ρ is induced by the ruling $\rho_1 : F_e \to \mathbb{P}^1$. Set $Q := \rho_1(P)$. By the construction $\rho^{-1}(Q) \cong Z \cup S_1$, i.e. the fiber of ρ over Q contains a point with multiplicity two and $a - 2$ points with multiplicity one. Hence there are no (C', ρ', ρ'') with C' a smooth curve, $\rho' : C \to C'$, $\rho'' : C' \to \mathbb{P}^1$, $\rho = \rho'' \circ \rho'$, $\deg(\rho') \geq 2$ and $\deg(\rho'') \geq 2$. □

Lemma 3. *Fix S, Y, C, u as in Lemma 1 and take any spanned line bundle L of degree > 0. Fix a general $A \in |L|$ and set $B := u(A)$. Then $S \cap B = \emptyset$ and $h^1(F_e, \mathcal{I}_{S\cup B}((a - 2)h + (ae - e - 2)f)) > 0$.*

Proof. Since $\deg(L) > 0$, $A \neq \emptyset$. Since L is spanned, $h^0(C, L(-Q)) = h^0(C, L) - 1$ for each $Q \in C$ and in particular for each $Q \in A$. Riemann–Roch gives $h^1(C, \mathcal{O}_C(A \setminus \{Q\}) = h^1(C, \mathcal{O}_C(A)$ for every $Q \in A$. Since $H^0(C, \omega_C) \cong H^0(F_e, \mathcal{I}_S((a - 2)h + (ae - e - 2)f))$, we get

$$h^0(F_e, \mathcal{I}_{S\cup(B\setminus\{P\}}((a - 2)h + (ae - e - 2)f)) \\ = h^0(F_e, \mathcal{I}_{S\cup B}((a - 2)h + (ae - e - 2)f)$$

for every $P \in B$. Hence $h^1(F_e, \mathcal{I}_{S\cup B}((a - 2)h + (ae - e - 2)f)) > 0$. □

Lemma 4. *Take e, a, x, S, Y, C as in Lemma 2. Then $d_1(C) = a$.*

Proof. The line bundle $u^*(\mathcal{O}_Y(f))$ gives $d_1(C) \leq a$. Assume $z := d_1(C) < a$ and take $L \in \mathrm{Pic}^z(C)$ evincing $d_1(C)$, i.e. evincing the gonality of C. Fix a general $A \in |L|$ and set $B := u(A)$. Lemma 3 gives

$$h^1(F_0, \mathcal{I}_{S\cup B}((a - 2)h + (ae - 2 - e)f)) > 0.$$

Since L is spanned and A is general, we have $S \cap B = B \cap h = \emptyset$. Lemma 2 gives $h^0(C, u^*(\mathcal{O}_Y(f))) = 2$. Let $v : C \to \mathbb{P}^1$ be the morphism induced by $|L|$ and $v' : C \to \mathbb{P}^1$ the morphism induced by $|u^*(\mathcal{O}_Y(f))|$. Since v' is not composed with an involution (Lemma 3), the induced map $(v, v') : C \to \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image. Hence for general B we have $\sharp(D \cap B) \leq 1$ for every $D \in |\mathcal{O}_{F_e}(f)|$. Since $h^0(F_e, \mathcal{O}_{F_e}(h + ef)) > z$, there is $A_1 \in |\mathcal{O}_{F_e}(h + ef)|$ containing B. Since $B \cap h = \emptyset$ and $\sharp(D \cap B) \leq 1$ for every $D \in |\mathcal{O}_{F_e}(f)|$, A_1 is irreducible. Hence $E \cong \mathbb{P}^1$. Since S is general and $h^0(F_e,\mathcal{O}_{F_e}(h+ef)) = e+2$, we have $\sharp(S\cap A_1)\le e+1$. Hence

$$\sharp(A_1\cap(S\cup B))\le z+e+1\le a+e.$$

Since $\deg(\mathcal{O}_{A_1}((a-2)h+(ae-e-2)f)) = ae-e-2\ge a+e-1$, we have

$$h^1(A_1,\mathcal{I}_{A_1\cap(S\cup B),A_1}((a-2)h+(ae-e-2)f)) = 0.$$

Hence the case $i=1$ of (1) gives

$$h^1(F_e,\mathcal{I}_{S\setminus S\cap A_1}((a-3)h+((a-1)e-e-2)f))>0.$$

Since $S\setminus S\setminus S\cap A_1$ is general and

$$x\le e(a-2)(ea-3e+2)/2\le h^0(F_e,\mathcal{O}_{F_e}((a-3)h+((a-1)e-e-2)f)),$$

we have

$$h^1(F_e,\mathcal{I}_{S\setminus S\cap A_1}((a-3)h+((a-1)e-e-2)f)) = 0,$$

a contradiction. □

Lemma 5. *Fix integers $e\ge 2$ and $a\ge 2$. Fix any integral $Y\in|\mathcal{O}_{F_e}(ah+eaf)|$ and call $u : C\to Y$ the normalization map. Then $s_{e+1+2j}(C)\le ae+je$ for every integer $j\ge 0$.*

Proof. We have $h^0(F_e,\mathcal{O}_{F_e}(h+(e+j)ef)) = e+2+2j$, for every integer $j\ge 0$. Since $a\ge 2$, we have $h^0(F_e,\mathcal{I}_Y(h+yf)) = 0$ for any y. We have $\mathcal{O}_{F_e}(h+(e+j)f)\cdot\mathcal{O}_{F_e}(ah+eaf) = a(e+j)$. Since for any $j\ge 0$ the linear system $|\mathcal{O}_{F_e}((h+(e+j)f)|$ embeds $F_e\setminus h$, the spanned line bundle $u^\ast(\mathcal{O}_Y((h+(e+j)f))$ gives $s_{e+1+2j}(C)\le ae+je$. □

Lemma 6. *Fix an integer $e\ge 2$. There is an integer $A_e\ge 5$ with the following property. Fix integers a,x such that $a\ge A_e$ and $0\le x\le ae-e-2$. Moreover, every base point free linear system on C with degree $\le ae$ and birationally very ample is induced (after deleting the base points) from a linear subspace of $H^0(F_e,\mathcal{O}_{F_e}(h+ef))$.*

Proof. Fix an integer $z\le ae$ such that there is a spanned $L\in\mathrm{Pic}^z(C)$ such that the morphism $v : C\to\mathbb{P}^k$, $k := h^0(C,L)-1$, induced by $|L|$ is birational onto its image. Fix a general $A\in|L|$ and set $B := u(A)$. Since L is spanned and A is general, we have $S\cap B=\emptyset$ and $B\cap h=\emptyset$. Lemma 3

$$h^1(F_0,\mathcal{I}_{S\cup B}((a-2)h+(ae-2-e)f))>0.$$

(a) Since the monodromy group G of the general hyperplane section of $v(C)$ is the full symmetric group S_z, B is in uniform position in F_e and in particular for all integers c,t such that $0\le c\le a$ and $t\ge ec$ and any $B'\subset B$, either $h^0(F_e,\mathcal{I}_{B'}(ch+tf)) = \max\{0,(c+1)(t+1)-\sharp(B')\}$ or $h^0(F_e,\mathcal{I}_B(ch+tf))>0$. In particular, $\sharp(D\cap B)\le 1$ for every $D\in|\mathcal{O}_{F_e}(f)|$.

(b) In this step we assume $h^0(F_e,\mathcal{I}_B(h+ef))>0$. Let t be the minimal non-negative integer such that $h^0(F_e,\mathcal{I}_B(h+tf))>0$. By assumption we have $t\le e$. Varying A in $|L|$, we get that $|L|$ is obtained (after deleting the base locus) from a linear subspace of $|\mathcal{O}_{F_e}(h+tf)|$. Since $|\mathcal{O}_{F_e}(h+tf)|$ sends $F_e \setminus h$ onto $\mathbb{P}^1$ if $t < e$, while v is birational onto its image, we get $t = e$. Since $h^0(F_e, \mathcal{I}_B(h+(e-1)f)) = 0$, step (a) gives $\sharp(D \cap B) \le e-1$ for every $\Gamma \in |\mathcal{I}_B(h+(e-1)f))|$. Since $\sharp(D\cap B) \le 1$ for every $D \in |\mathcal{O}_{F_e}(1)|$ and $z > e$, T is irreducible. Hence $T \cong \mathbb{P}^1$. Since $\sharp(B) \le Y \cdot T = ae$, we have $z \le ae$ and if inequality holds, then $|L|$ is induced without deleting any base point from $|\mathcal{O}_{F_e}(h+ef)|$. Hence $k \le e+1$ and v is induced (after deleting the base points) from a linear subspace of $H^0(F_e, \mathcal{O}_{F_e}(h+ef))$. We get that if L evinces $s_{e+1}(C)$ and the assumption of this step holds, then $s_{e+1}(C) = ae$ and $L \cong u^*(\mathcal{O}_Y(h+ef))$.

(c) From now on we assume $h^0(F_e, \mathcal{I}_B(h+ef)) = 0$. To conclude the proof of the lemma it is sufficient to find a contradiction for $a \gg 0$ and any $x \le ae-e-2$. Set $c := \lfloor z/(e+1)\rfloor$. Set $S_0 := S$ and $B_0 := B$. Fix $A_1 \in |\mathcal{O}_{F_e}(h+ef)|$ such that $a_1 := \sharp(A_1 \cap B_0)$ is maximal. Set $S_1 := S_0 \setminus S_0 \cap A_1$ and $B_1 := B_0 \setminus B_0 \cap A_1$. For each integer $i \ge 2$ define recursively the curve $A_i \in |\mathcal{O}_{F_e}(h+ef)|$, the integer a_i, and the sets S_i, B_i in the following way. Fix $A_i \in |\mathcal{O}_{F_e}(h+ef)|$ such that $a_i := \sharp(A_i \cap B_{i-1})$ is maximal. Set $S_i := S_{i-1} \setminus S_{i-1} \cap A_i$ and $B_i := B_{i-1} \setminus B_{i-1} \cap A_i$. Since $h^0(F_e, \mathcal{O}_{F_e}(h+ef)) = e+2$ and $h^0(F_e, \mathcal{I}_B(h+ef)) = 0$, step (a) gives $a_i \le e+1$ for all i. Since $h^0(F_e, \mathcal{O}_{F_e}(h+ef)) = e+2$ and a_i is maximal, either $a_i = e+1$ or $B_i = \emptyset$. Hence $a_i = e+1$ for $i \le c$, $a_{c+1} = z - c(a+1) \le e+1$ and $a_i = 0$ for all $i \ge c+2$. Assume $a \ge 4e$. Hence $(e+1)^2(a-3) \ge e(e+2)a$. Since $z \le ea$, we get $c \le a-4$. For each integer $i = 1, \dots, c+1$ we have an exact sequence

$$
\begin{aligned}
&0 \to \mathcal{I}_{S_i\cup B_i}((a-2-i)f + (e(a-i)-e-2)f)\\
(2)\qquad &\to \mathcal{I}_{S_{i-1}\cup B_{i-1}}((a-1-i)h + (e(a-i+1)-e-2)f)\\
&\to \mathcal{I}_{A_i\cap(S_{i-1}\cup B_{i-1},A_i}((a-1-i)h + (e(a-i+1)-e-2)f) \to 0.
\end{aligned}
$$

Fix $i \in \{1, \dots, c\}$. By step (a) we have $\sharp(D\cap B) \le 1$ for every $D \in |\mathcal{O}_{F_e}(f)|$. Hence A_i is irreducible. Hence $A_i \cong \mathbb{P}^1$. Since $\sharp(D \cap B) \le 1$ for every $B \in |\mathcal{O}_{F_e}(f)|$ and $B \cap h = \emptyset$, even if $a_{c+1} \le a$ we may take an irreducible $A_{c+1} \in |\mathcal{O}_{F_e}(f)|$ containing B_{c+1}. Assume for a moment $c+1 \le a-5$. Since $e \ge 2$, we have $e(a-c+1)-e-2 \ge 2e+1$. Set $x_i := \sharp(S_{i-1} \cap A_i)$. Since S is general, we have $x_i \le e+1$. Hence $x_i + a_i \le 2e+2$. Since $A_i \cong \mathbb{P}^1$ and

$$
\begin{aligned}
\deg(\mathcal{O}_{A_i}((a-1-i)h + (e(a-i+1)-e-2)f)) &= e(a-i+1)-e-2)\\
\ge e(a-c+1)-e-2 \ge 2e+1,&
\end{aligned}
$$

we have

$$h^1(A_i, \mathcal{I}_{A_i\cap(S_{i-1}\cup B_{i-1},A_i}((a-1-i)h + (e(a-i+1)-e-2)f)) = 0.$$

Hence applying (2) first for $i = 1$, then for $i = 2$, and so on up to $i = c+1$, we get

$$h^1(F_e, \mathcal{I}_{S_{c+1}}((a-3-c)f + (e(a-c-1)-e-2)f)) > 0.$$

Since $2e \geq e+1$, we have

$$h^1(F_e, \mathcal{O}_{F_e}((a-3-c)f + (e(a-c-1)-e-2)f)) = 0.$$

Since S is general and $S_c \subseteq S$, to have $h^1(F_e, \mathcal{I}_{S_{c+1}}((a-3-c)f + (e(a-c-1)-e-2)f)) = 0$ (and hence a contradiction), it is sufficient to have

$$\sharp(S_c) \leq h^0(F_e, \mathcal{O}_{F_e}((a-3-c)f + (e(a-c-1)-e-2)f)).$$

Since $\sharp(S_c) \leq x$, it is sufficient to have $x \leq (a-3-c)(e(a-3-c)+2e-2)/2$. Since $x \leq ae-e-2$, it is sufficient to have $(a-c-3)^2 e/2 \geq ae$. Thus it is sufficient to have $c \leq a-3-\sqrt{2a}$. Since $c \leq ea/(e+1)$, it is sufficient to have $a-(e+1)\sqrt{2a}-3e-3 \geq 0$. Hence we may take $A_e = 32(e+1)^2$. Notice that we also checked the assumption $a-c-1 \leq a-5$. □

Lemma 7. *Take $e \geq 2$, A_e, $a \geq A_e$, $0 \leq x \leq ea-e-2$, S, Y and C as in Lemma 5.*

(a) *We have $s_e(C) = ea-1-\min\{1, x\}$.*

(b) *If $x > 0$, then each $L \in Pic(C)$ evincing $s_e(C)$ is induced by $|\mathcal{I}_{\{P\}}(h+ef)|$ (after deleting the degree 2 base locus $u^{-1}(P)$) for some $P \in S$. For an arbitrary x any spanned and birationally very ample line bundle M of degree $ea-1$ is induced by $|\mathcal{I}_{\{P\}}(h+ef)|$ (after deleting the degree 1 base locus $u^{-1}(P)$) for some $P \in Y \setminus (S \cup h)$.*

Proof. The linear systems described in part (b) shows that $s_e(C) \leq ea-1-\min\{1, x\}$. By Lemma 7 any such birationally very ample and spanned complete linear system $|L|$ is induced (after deleting the base locus) from a codimension 1 linear subspace V of $H^0(F_e, \mathcal{O}_{F_e}(h+ef))$. Call $\mathcal{B} \subset F_e$ the base scheme of V as a linear system on F_e and $\mathbb{B}$ the base locus of $u^*(V)$ on C. Since $h^0(C, u^*(\mathcal{O}_Y(h+ef))) \geq e+2$, we have $\mathbb{B} \neq \emptyset$. Obviously $\mathbb{B}_{red} = u^{-1}(\mathcal{B} \cap Y)$. Hence $\mathcal{B} \cap Y \neq \emptyset$. Since $\mathcal{O}_h(h+ef) \cong \mathcal{O}_h$,

$$h^0(F_e, \mathcal{O}_{F_e}(h+ef)) = 2 + h^0(F_e, \mathcal{O}_{F_e}(ef))$$

and V has codimension 1 in $H^0(F_e, \mathcal{O}_{F_e}(h+ef))$, we have $h \cap \mathcal{B} = \emptyset$. Since $|\mathcal{O}_{F_e}(h+ef)|$ induces an embedding of $F_e \setminus h$, the scheme $\mathcal{B}$ must be a single point, P, with its reduced structure. Since $\mathcal{B} \cap Y \neq \emptyset$, we have $P \in Y$. We have $\deg(L) = ae-1$ if $P \notin S$ and $\deg(L) = ae-2$ if $P \in S$. □

3. Proof of Theorem 1. We fix the integer $r \geq 3$ for which we want to prove Theorem 1 and set $e := r-1$. Hence $e \geq 2$. Fix A_e as in Lemma 6 and any integer $g \geq eA_e^2/2 - eA_e + e + 2$. Let a be the minimal integer such that $g \leq g_{a,e}$. Since $g_{a,e} - g_{a-1,e} = ae-e-1$, we have $a \geq A_e$ and there is a unique integer x such that $0 \leq x \leq ae-e-2$ and $g = g_{a,e} - x$. Take C as in Lemmas 6 and 7. Lemma 6 gives $s_{e+1}(C)) = ae$. Hence it is sufficient to prove that $s_{e+2}(C) > (e+2)ea/(e+1)$. Assume $z := s_{e+2}(C) \leq (e+2)ea/(e+1)$ and fix $L \in \mathrm{Pic}^z(C)$ evincing $s_{e+2}(C)$. The line bundle L is spanned, $h^0(C, L) = e+3$ and $|L|$ induces a morphism

$v : C \to \mathbb{P}^{e+2}$ birationally onto its image and with $v(C)$ a degree z non-degenerate curve with arithmetic genus $\geq g$. Set $m_1 := \lfloor (z-1)/(e+2) \rfloor$, $\epsilon_1 = z - 1 - m_1(e+2)$, $\mu_1 := 0$ if $\epsilon_1 \neq e+1$ and $\mu_1 := 1$ if $\epsilon_1 = e+1$. Set $\pi_1(z, e+2) = (e+2)m_1(m_1-1)/2 + m_1(\epsilon_1+1) + \mu_1$. Notice that

$$\pi_1(z, e+2) \leq z(z+2)/2(e+2) \leq ea(e+2)(ea(e+2)+2e+2)/(2(e+2)(e+1)^2).$$

Notice that $e^2(e+2)^2/(2(e+2)(e+1)^2) < e/2$. Since $g > g_{a-1,e} = 1 + (a-1)(ae-2-2e)/2$, we have $g > \pi(z, e+2)$ if $a \gg 0$, say if $a \geq A'_e$. Hence [3], Theorem 3.15, gives that $v(C)$ is contained in a degree $e+1$ surface $T \subset \mathbb{P}^{e+2}$. By the classification of all minimal degree surfaces ([2]), either T is a cone over a rational normal curve or $T \cong F_m$ embedded by the complete linear system $|\mathcal{O}_{F_{e+1}}(h + (e+1+m)f)|$ for some integer $m \equiv e+1 \pmod 2$ with $0 \leq m \leq e-1$. In the latter case we set $E := v(C)$. In the former case T is the image of F_{e+1} by the complete linear system $|\mathcal{O}_{F_{e+1}}(h + (e+1)f)|$; in this case set $m := e+1$ and call E the strict transform of $v(C)$ in F_{e+1}. In both cases E is a curve contained in F_m with C as its normalization. Call $u' : C \to E$ the normalization map. Hence there are integers c, y such that $E \in |\mathcal{O}_{F_m}(ch + yf)|$ with $y \geq mc$ and $c > 0$. Lemma 4 gives $c \geq a$; if $m = 0$ it also gives $y \geq a$.

(a) Here we assume $m \leq e-1$. Let $T' \subset \mathbb{P}^e$ be the image of F_m by the complete linear system $|\mathcal{O}_{F_m}(h + (e+m)f)|$. Since either $T' \cong F_m$ (case $m \neq e-1$) or T' is the blowing down of h (case $m = e-1$), the image of E in T' gives $s_e(C) \leq \mathcal{O}_{F_m}(h + (e+m)f) \cdot \mathcal{O}_{F_m}(ch + yf) = z - c$. Since $c \geq a$, Lemma 7 gives $z \geq c + ae - 2 \geq a(e+1) - 2$, contradicting the assumption $z \leq ea(e+2)/(e+1)$ (with $a > 2(e+1)^2$).

(b) Now assume $m = e+1$. Since $y \geq mc = (e+1)c$ and $c \geq a$ (Lemma 6), this case is impossible.

The proof of Theorem 1 is complete.

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