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Weighted sub-Bergman Hilbert spaces

Abstract. We consider Hilbert spaces which are counterparts of the de Branges–Rovnyak spaces in the context of the weighted Bergman spaces A^2_α, $-1 < \alpha < \infty$. These spaces have already been studied in [8], [7], [5] and [1]. We extend some results from these papers.

1. Introduction. Let $\mathbb{D}$ denote the unit disk in the complex plane. For $-1 < \alpha < \infty$, the weighted Bergman space A^2_α is the space of holomorphic functions f in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < \infty,$$

where

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \frac{dxdy}{\pi} = (\alpha+1)(1-|z|^2)^\alpha dA(z), \quad z = x+iy.$$

The space A^2_α is a Hilbert space with the inner product $\langle f, g\rangle_\alpha$ inherited from $L^2(\mathbb{D}, dA_\alpha)$. It then follows that if

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$$

are functions in A^2_α, then

$$\langle f, g\rangle_\alpha = \sum_{n=0}^{\infty} \frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} \hat{f}(n)\overline{\hat{g}(n)}.$$

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Clearly, $A_0^2 = A^2$ is the Bergman space on the unit disk.

For $\varphi \in L^\infty(\mathbb{D})$ the Toeplitz operator T_φ^α on A_α^2 is defined by

$$T_\varphi^\alpha(f) = P_\alpha(\varphi f), \quad f \in A_\alpha^2,$$

where $P_\alpha : L^2(\mathbb{D}, dA_\alpha) \to A_\alpha^2$ is the projection operator

$$P_\alpha(f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \bar{w}z)^{\alpha+2}} dA_\alpha(w).$$

Suppose that T is a contraction on a Hilbert space H. Following [4], we define the space $\mathcal{H}(T)$ to be the range of the operator $(I - TT^*)^{1/2}$ with the inner product given by

$$\left\langle (I - TT^*)^{1/2} f, (I - TT^*)^{1/2} g \right\rangle_{\mathcal{H}(T)} = \langle f, g \rangle, \quad f, g \in (\ker(I - TT^*)^{1/2})^\perp.$$

For φ in the closed unit ball of H^∞, the spaces $\mathcal{H}(T_\varphi^\alpha)$ and $\mathcal{H}(T_{\overline{\varphi}}^\alpha)$ are denoted by $\mathcal{H}_\alpha(\varphi)$ and $\mathcal{H}_\alpha(\overline{\varphi})$, respectively. For the case when $\alpha = 0$ these spaces were studied by Kehe Zhu in [7], [8]. He proved that the spaces $\mathcal{H}_0(\varphi)$ and $\mathcal{H}_0(\overline{\varphi})$ coincide as sets and both the spaces contain H^∞. Zhu also proved that if φ is a finite Blaschke product B, then, as sets, $\mathcal{H}_0(B) = \mathcal{H}_0(\bar{B}) = H^2$, the Hardy space on the unit disk. These results were extended to positive α in [5], where the author proved that

$$\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\bar{B}) = A_{\alpha-1}^2.$$

For α as above, we define the space $\mathcal{D}(\alpha)$ to be the set of holomorphic functions in $\mathbb{D}$ and such that $f' \in L^2(\mathbb{D}, dA_\alpha)$. Here we further extend the above-mentioned result and show that for $-1 < \alpha < \infty$,

$$\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\bar{B}) = D(\alpha + 1) \quad \text{as sets.}$$

After sending this paper for publication we found that a different proof of these equalities was given by F. Symesak in [6].

For $a \in \mathbb{D}$, set

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

Let $K_a^\alpha(z) = \frac{1}{(1-\bar{a}z)^{\alpha+2}}$ be a reproducing kernel for A_α^2 and let

$$k_a^\alpha(z) = \frac{(1 - |a|^2)^{1+\frac{\alpha}{2}}}{(1 - \bar{a}z)^{\alpha+2}}$$

be the normalized kernel. Since the linear operator $A : A_\alpha^2 \to A_\alpha^2$ defined by

$$Af(z) = k_a^\alpha f \circ \varphi_a$$

is a surjective isometry, the functions

$$e_{a,n} = \frac{k_a^\alpha \varphi_a^n}{\sqrt{(\alpha + 1)\beta(n + 1, \alpha + 1)}}$$

form an orthonormal basis for A_α^2.

The following formula for the operator $(I - T^\alpha_{\overline{\varphi_a}} T^\alpha_{\varphi_a})^{1/2} = (T^\alpha_{1-|\varphi_a|^2})^{1/2}$ has been derived in [5]:

$$(T^\alpha_{1-|\varphi_a|^2})^{1/2} = \sum_{n=0}^{\infty} \frac{\sqrt{\alpha+1}}{\sqrt{n+\alpha+2}} e_{a,n} \otimes e_{a,n},$$

where $e_{a,n} \otimes e_{a,n}(f) = \langle f, e_{a,n}\rangle_\alpha e_{a,n}$ for $f \in A^2_\alpha$.

In this paper we obtain the analogous formula for the operator $(I - T^\alpha_{\varphi_a} T^\alpha_{\overline{\varphi_a}})^{1/2}$. We also find the formulas for the inner products in $\mathcal{H}_\alpha(\varphi_a)$ and $\mathcal{H}_\alpha(\overline{\varphi_a})$ in terms of the Fourier coefficients with respect to the orthonormal basis $\{e_{a,n}\}$.

We note that since

$$\varphi_a^n(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a^{n-k} \frac{(1-|a|^2)^k z^k}{(1-\bar{a}z)^k}$$

(see [5]), we have

$$\begin{aligned}\langle f, \varphi_a^n K_a^\alpha\rangle_\alpha &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \bar{a}^{n-k} (1-|a|^2)^k \left\langle f, \frac{z^k}{(1-\bar{a}z)^{k+\alpha+2}} \right\rangle_\alpha \\ &= \bar{a}^n f(a) + \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k \bar{a}^{n-k} (1-|a|^2)^k f^{(k)}(a)}{(\alpha+2)(\alpha+3)\dots(\alpha+k+1)}.\end{aligned}$$

So, in particular, the constant function $f_1 \equiv 1$ can be written as follows

$$\begin{aligned}1 \equiv f_1 &= \sum_{n=0}^{\infty} \frac{\bar{a}^n}{\|\varphi_a^n K_a^\alpha\|} e_{a,n}(z) = \sum_{n=0}^{\infty} \frac{\bar{a}^n (1-|a|^2)^{\frac{\alpha}{2}+1}}{\sqrt{(\alpha+1)\beta(n+1,\alpha+1)}} e_{a,n} \\ &= \frac{(1-|a|^2)^{\alpha+2}}{(1-\bar{a}z)^{\alpha+2}} \sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(\alpha+2)} \bar{a}^n \left(\frac{z-a}{1-\bar{a}z}\right)^n.\end{aligned}$$

2. The spaces $\boldsymbol{H_\alpha(\varphi_a)}$ and $\boldsymbol{\mathcal{H}_\alpha(\overline{\varphi_a})}$. The following theorem describes the operator $(I - T^\alpha_{\varphi_a} T^\alpha_{\overline{\varphi_a}})^{\frac{1}{2}}$.

Theorem 2.1. *For $a \in \mathbb{D}$,*

$$(I - T^\alpha_{\varphi_a} T^\alpha_{\overline{\varphi_a}})^{\frac{1}{2}} = \sum_{n=0}^{\infty} \sqrt{\frac{\alpha+1}{n+\alpha+1}}\, e_{a,n} \otimes e_{a,n}.$$

Proof. Our aim is to prove that the functions $\varphi_a^n K_a^\alpha$, $n = 0, 1\dots$, are eigenvectors of the operator $(I - T^\alpha_{\varphi_a} T^\alpha_{\overline{\varphi_a}})^{\frac{1}{2}}$ with corresponding eigenvalues

$\sqrt{\frac{\alpha+1}{n+\alpha+1}}$. We have

$$\begin{aligned}
T^{\alpha}_{\overline{\varphi_a}}(\varphi_a^n K_a^\alpha)(z) &= \int_{\mathbb{D}} \frac{\overline{\varphi_a(w)}\varphi_a^n(w)}{(1-\bar{a}w)^{\alpha+2}(1-z\bar{w})^{\alpha+2}}\, dA_\alpha(w)\\
&= \int_{\mathbb{D}} \frac{\bar{u}u^n}{(1-\bar{u}a-z\bar{a}+z\bar{u})^{2+\alpha}}\, dA_\alpha(u)\\
&= K_a^\alpha(z)\int_{\mathbb{D}} \frac{\bar{u}u^n}{(1-\bar{u}\varphi_a(z))^{2+\alpha}}\, dA_\alpha(u)\\
&= K_a^\alpha(z)\int_{\mathbb{D}} \sum_{k=0}^{\infty}\frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)}(\bar{u}\varphi_a(z))^k\bar{u}u^n\, dA_\alpha(u)\\
&= \frac{\Gamma(n+1+\alpha)}{(n-1)!\Gamma(2+\alpha)}K_a^\alpha(z)\varphi_a^{n-1}(z)\int_{\mathbb{D}} |u|^{2n}\, dA_\alpha(u)\\
&= \frac{n}{n+1+\alpha}K_a^\alpha(z)\varphi_a^{n-1}(z).
\end{aligned}$$

Hence

$$(I-T^\alpha_{\varphi_a}T^\alpha_{\overline{\varphi_a}})(\varphi_a^n K_a^\alpha)(z) = \frac{\alpha+1}{n+\alpha+1}\varphi_a^n K_a^\alpha,$$

and consequently,

$$(I-T^\alpha_{\varphi_a}T^\alpha_{\overline{\varphi_a}})^{\frac{1}{2}}(\varphi_a^n K_a^\alpha)(z) = \sqrt{\frac{\alpha+1}{n+\alpha+1}}\varphi_a^n K_a^\alpha.$$

Expanding $f \in A^2_\alpha$ in the Fourier series with respect to the basis $\{e_{a,n}\}$

$$f = \sum_{n=0}^{\infty}\langle f, e_{a,n}\rangle e_{a,n},$$

we find that

$$\begin{aligned}
\left(I-T^\alpha_{\varphi_a}T^\alpha_{\overline{\varphi_a}}\right)^{\frac{1}{2}} f &= \sum_{n=0}^{\infty}\langle f, e_{a,n}\rangle \left(I-T^\alpha_{\varphi_a}T^\alpha_{\overline{\varphi_a}}\right)^{\frac{1}{2}} e_{a,n}\\
&= \sum_{n=0}^{\infty}\langle f, e_{a,n}\rangle \sqrt{\frac{\alpha+1}{n+\alpha+1}}e_{a,n}\\
&= \sum_{n=0}^{\infty}\sqrt{\frac{\alpha+1}{n+\alpha+1}}(e_{a,n}\otimes e_{a,n})f.
\end{aligned}$$

□

By Proposition 1.3.10 in [9] we also get

Corollary 2.1. $(I-T^\alpha_{\varphi_a}T^\alpha_{\overline{\varphi_a}})^{\frac{1}{2}}$ *is a compact operator on* A^2_α.

In our next result we give formulas for inner products $\langle f, g\rangle_{\mathcal{H}_\alpha(\varphi_a)}$ and $\langle f, g\rangle_{\mathcal{H}_\alpha(\overline{\varphi_a})}$ in terms of the Fourier coefficients $\hat{f}_a(n) = \langle f, e_{a,n}\rangle_\alpha$ and $\hat{g}_a(n) = \langle f, e_{a,n}\rangle_\alpha$.

Proposition 2.1. *For $a \in \mathbb{D}$,*

$$\langle f, g\rangle_{\mathcal{H}_\alpha(\varphi_a)} = \langle f, g\rangle_\alpha + \sum_{n=1}^{\infty} \frac{n}{\alpha+1} \hat{f}_a(n)\overline{\hat{g}_a(n)}$$

and

$$\langle f, g\rangle_{\mathcal{H}_\alpha(\overline{\varphi_a})} = \langle f, g\rangle_\alpha + \sum_{n=0}^{\infty} \frac{n+1}{\alpha+1} \hat{f}_a(n)\overline{\hat{g}_a(n)}.$$

Proof. We shall prove the first formula. The other can be proved analogously. By Sarason ([4], p. 3) we know that $f,\ g \in \mathcal{H}_\alpha(\varphi_a)$ if and only if $T^\alpha_{\overline{\varphi_a}} f \in \mathcal{H}_\alpha(\overline{\varphi_a})$ and

$$\langle f, g\rangle_{\mathcal{H}_\alpha(\varphi_a)} = \langle f, g\rangle_\alpha + \langle T^\alpha_{\overline{\varphi_a}} f, T^\alpha_{\overline{\varphi_a}} g\rangle_{\mathcal{H}_\alpha(\overline{\varphi_a})}.$$

It follows from the proof of Theorem 2.1 that

$$T^\alpha_{\overline{\varphi_a}}(\varphi_a^n K_a^\alpha)(z) = \frac{n}{n+1+\alpha} K_a^\alpha(z)\varphi_a^{n-1}(z)$$

and consequently,

$$T^\alpha_{\overline{\varphi_a}}(e_{a,n}) = \sqrt{\frac{n}{n+1+\alpha}}\, e_{a,n-1}.$$

Hence

$$\langle T^\alpha_{\overline{\varphi_a}} f, T^\alpha_{\overline{\varphi_a}} g\rangle_{\mathcal{H}_\alpha(\overline{\varphi_a})} = \sum_{n=1}^{\infty} \frac{n}{n+1+\alpha} \hat{f}_a(n)\overline{\hat{g}_a(n)} \|e_{a,n-1}\|^2_{\mathcal{H}_\alpha(\overline{\varphi_a})}.$$

Since

$$\left(I - T^\alpha_{\overline{\varphi_a}} T^\alpha_{\varphi_a}\right)^{\frac{1}{2}} (e_{a,n}) = \sqrt{\frac{\alpha+1}{n+\alpha+2}}\, e_{a,n},$$

we have

$$\|e_{a,n-1}\|^2_{\mathcal{H}_\alpha(\overline{\varphi_a})} = \frac{n+1+\alpha}{\alpha+1}.$$

□

3. Finite Blaschke products. Throughout this section B will stand for a finite Blaschke product. The spaces $\mathcal{H}_\alpha(B)$ and $\mathcal{H}_\alpha(\overline{B})$ have been described for $\alpha \geq 0$ in [8] and [1]. We will use the methods developed in these papers to extend the result for $-1 < \alpha < 0$.

For $-1 < \alpha < \infty$ let $\mathcal{D}(\alpha)$ denote the Hilbert space consisting of analytic functions in $\mathbb{D}$ whose derivatives are in $L^2(\mathbb{D}, dA_\alpha)$ with the inner product

$$\langle f, g\rangle_{\mathcal{D}(\alpha)} = \hat{f}(0)\overline{\hat{g}(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\, dA_\alpha(z).$$

We shall show the following

Theorem 3.1. *For* $-1<\alpha<\infty$,

$$\mathcal{H}_\alpha(\overline{B})=\mathcal{D}(\alpha+1)$$

as sets.

Proof. As in [7] and [1] we define the Hilbert space $A^2_{\alpha,B}$ consisting of functions f analytic in $\mathbb{D}$ and such that

$$\int\limits_{\mathbb{D}}|f(z)|^2(1-|B(z)|^2)\,dA_\alpha(z)<\infty$$

with the inner product

$$\langle f,g\rangle_{A^2_{\alpha,B}}=\int\limits_{\mathbb{D}}f(z)\overline{g(z)}(1-|B(z)|^2)\,dA_\alpha(z).$$

Since, for $z\in\mathbb{D}$,

$$1-|B(z)^2|\sim 1-|z|^2 \qquad \text{(see, e.g., Lemma 1 of [8])},$$

the function $g\in A^2_{\alpha,B}$ if and only $g\in A^2_{\alpha+1}$ and the norms in these spaces are equivalent.

It was proved in [8] and [1] that the space $\mathcal{H}_\alpha(\overline{B})$ consists of analytic functions of the form

$$f(z)=S_\alpha(g)(z)=\int\limits_{\mathbb{D}}\frac{1-|B(w)|^2}{(1-z\bar{w})^{\alpha+2}}g(w)\,dA_\alpha(w), \tag{3.1}$$

where $g\in A^2_{\alpha,B}$. It then follows that if $f\in\mathcal{H}_\alpha(\overline{B})$, then

$$f'(z)=(\alpha+2)\int\limits_{\mathbb{D}}\frac{\bar{w}(1-|B(w)|^2)}{(1-z\bar{w})^{\alpha+3}}g(w)\,dA_\alpha(w).$$

By Theorem 1.9 of [3] the operator

$$\Lambda g(z)=\int\limits_{\mathbb{D}}\frac{(1-|w|^2)^{\alpha+1}}{|1-z\bar{w}|^{\alpha+3}}|g(w)|\,dA(w)$$

is bounded on $L^2(\mathbb{D},dA^2_{\alpha+1})$. Therefore, there is a constant $C>0$ such that

$$\int\limits_{\mathbb{D}}|f'(z)|^2\,dA_{\alpha+1}(z)\le\|\Lambda g\|_{L^2(\mathbb{D},dA^2_{\alpha+1})}\le C\|g\|_{A^2_{\alpha+1}},$$

which proves the inclusion $\mathcal{H}_\alpha(\overline{B})\subset\mathcal{D}(\alpha+1)$. To prove that $\mathcal{D}(\alpha+1)\subset\mathcal{H}_\alpha(\overline{B})$ we consider the operator $R_\alpha:\mathcal{D}(\alpha+1)\to A^2_{\alpha,B}$ given by

$$R_\alpha f(z)=(\alpha+2)zf'(z)+f(0).$$

Using the Fubini Theorem, one can easily check that $R_\alpha = S_\alpha^*$, where $S_\alpha : A^2_{\alpha,B} \to \mathcal{D}(\alpha+1)$ is given by (3.1). Indeed, for $f \in \mathcal{D}(\alpha+1)$,

$$\begin{aligned}
\langle f, S_\alpha g\rangle_{\mathcal{D}(\alpha+1)} &= \hat{f}(0)\overline{\widehat{S_\alpha g}(0)} \\
&\quad + (\alpha+2)\int_{\mathbb{D}} f'(z) \int_{\mathbb{D}} \frac{(1-|B(w)|^2)w}{(1-\bar{z}w)^{\alpha+3}}\overline{g(w)} dA_\alpha(w)\, dA_{\alpha+1}(z) \\
&= \hat{f}(0)\langle 1, g\rangle_{A^2_{\alpha,B}} \\
&\quad + \int_{\mathbb{D}} (1-|B(w)|^2) w\overline{g(w)}(\alpha+2) f'(w)\, dA_\alpha(w) \\
&= \langle R_\alpha f, g\rangle_{A^2_{\alpha,B}}.
\end{aligned}$$

Since R_α is invertible, the image of the unit ball of $\mathcal{D}(\alpha+1)$ under R_α contains a ball of radius $r > 0$ centered at zero. As in [8], [1], for every unit vector $g \in A^2_{\alpha,B}$ we have

$$\begin{aligned}
\|S_\alpha g\|_{\mathcal{D}(\alpha+1)} &= \sup\left\{\left|\langle S_\alpha g, f\rangle_{\mathcal{D}(\alpha+1)}\right| : \|f\|_{\mathcal{D}(\alpha+1)} \le 1\right\} \\
&= \sup\left\{\left|\langle g, R_\alpha f\rangle_{A^2_{\alpha,B}}\right| : \|f\|_{\mathcal{D}(\alpha+1)} \le 1\right\} \\
&= \sup\left\{\left|\int_{\mathbb{D}} g(w)\overline{R_\alpha f(w)}(1-|B(w)|^2)\, dA_\alpha(w)\right| : \|f\|_{\mathcal{D}(\alpha+1)} \le 1\right\} \\
&\ge \sup\left\{\left|\int_{\mathbb{D}} g(w)\overline{h(w)}(1-|B(w)|^2)\, dA_\alpha(w)\right| : \|h\|_{A^2_{\alpha,B}} \le r\right\} \\
&= r\|g\|_{A^2_{\alpha,B}} = r.
\end{aligned}$$

This means that S_α is bounded from below, so that its range is closed in $\mathcal{D}(\alpha+1)$. Since polynomials are dense in the space $\mathcal{D}(\alpha+1)$, it is enough to prove that $S_\alpha(A^2_{\alpha,B})$ contains all polynomials. To show that z^n is in $S_\alpha(A^2_{\alpha,B})$ consider the closed subspace M of $A^2_{\alpha,B}$ spanned by functions z^m, $m \neq n$, $m \in \mathbb{N}$. Let g be a unit vector in $A^2_{\alpha,B} \ominus M$. Then

$$S_\alpha(g)(z) = \int_{\mathbb{D}} \frac{1-|B(u)|^2}{(1-z\bar{u})^{\alpha+2}} g(u)\, dA_\alpha(u) = \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} z^n \langle g, u^n\rangle_{A^2_{\alpha,B}}$$

for every $z \in \mathbb{D}$. If $\langle g, u^n\rangle_{A^2_{\alpha,B}} = 0$ for every unit vector g in $A^2_{\alpha,B} \ominus M$, then it will follow that $z^n \in M$, which is clearly impossible. So, there is $c_n \neq 0$ such that $c_n z^n \in S_\alpha(A^2_{\alpha,B})$. □

We remark that also in the case when $-1 < \alpha < 0$, $\mathcal{H}_\alpha(B) = H_\alpha(\overline{B})$. It follows from Douglas criterion that $H_\alpha(\overline{B}) \subset H_\alpha(B)$ (see [4]). Moreover, it was showed in [5] that for $-1 < \alpha < 0$, $\mathcal{H}_\alpha(B)$ is equal to a Hilbert space with the reproducing kernel $K_w^\alpha(z) = (1 - \bar{w}z)^{-(1+\alpha)}$. It is easy to see that the norm in such a space is given by

$$\|f\|_\alpha^2 = \frac{1}{(\alpha+1)(\alpha+2)}\|f'\|^2_{A^2_{\alpha+1}} + \|f\|^2_{A_\alpha}. \tag{3.2}$$

Indeed, for $z, w \in \mathbb{D}$ we have

$$K_w^\alpha(z) = k^\alpha(\bar{w}z)$$

where

$$k^\alpha(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1+\alpha)}{k!\Gamma(1+\alpha)}(\bar{w}z)^k.$$

This means that this space is the weighted Hardy space introduced in [2] with the generating function k^α. Hence

$$\|z^k\|^2 = \frac{k!\Gamma(\alpha+2)}{\Gamma(k+\alpha+2)}$$

and formula (3.2) follows. Thus, also for $-1 < \alpha < 0$, $\mathcal{H}_\alpha(B) = \mathcal{D}(\alpha+1)$ $= \mathcal{H}_\alpha(\overline{B})$. Finally, we note that in this case H^∞ is not contained in $\mathcal{H}_\alpha(B) = \mathcal{H}_\alpha(\overline{B})$. This follows, for example, from the result proved in [10] that H^∞ is contained in the weighted Hardy space $H^2(\beta)$ if and only if β is bounded.

References

[1] Abkar, A., Jafarzadeh, B., *Weighted sub-Bergman Hilbert spaces in the unit disk*, Czechoslovak Math. J. **60** (2010), 435–443.

[2] Cowen, C., MacCluer, B., *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.

[3] Hedenmalm, H., Korenblum, B., Zhu, K., *Theory of Bergman Spaces*, Spinger-Verlag, New York, 2000.

[4] Sarason, D., *Sub-Hardy Hilbert Spaces in the Unit Disk*, Wiley, New York, 1994.

[5] Sultanic, S., *Sub-Bergman Hilbert spaces*, J. Math. Anal. Appl. **324** (2006), 639–649.

[6] Symesak, F., *Sub-Bergman spaces in the unit ball of* $\mathbb{C}^n$, Proc. Amer. Math. Soc. **138** (2010), 4405–4411.

[7] Zhu, K., *Sub-Bergman Hilbert spaces in the unit disk*, Indiana Univ. Math. J. **45** (1996), 165–176.

[8] Zhu, K., *Sub-Bergman Hilbert spaces in the unit disk, II*, J. Funct. Anal. **202** (2003), 327–341.

[9] Zhu, K., *Operator Theory in Function Spaces*, Dekker, New York, 1990.

[10] Zorboska, N., *Composition operators induced by functons with supremum strictly smaller than 1*, Proc. Amer. Math. Soc. **106** (1989), 679–684.

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