doi: 10.1515/umcsmath-2015-0012

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXIX, NO. 1, 2015 SECTIO A 59–71

GRZEGORZ DYMEK

On pseudo-BCI-algebras

Abstract. The notion of normal pseudo-BCI-algebras is studied and some characterizations of it are given. Extensions of pseudo-BCI-algebras are also considered.

1. Introduction. Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are (pseudo-)MV-algebras, (pseudo-)BL-algebras, (pseudo-)BCK-algebras, (pseudo-)BCI-algebras and others. They are strongly connected with logic. For example, BCI-algebras introduced in [8] have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming.

The notion of pseudo-BCI-algebras has been introduced in [1] as an extension of BCI-algebras. Pseudo-BCI-algebras are algebraic models of some extension of a non-commutative version of the BCI-logic (see [5] for details). These algebras have also connections with other algebras of logic such as pseudo-BCK-algebras, pseudo-BL-algebras and pseudo-MV-algebras. More about those algebras the reader can find in [7].

The paper is devoted to pseudo-BCI-algebras. In Section 2 we give the necessary material needed in the sequel and also some new results concerning p-semisimple part and branches of pseudo-BCI-algebras. In Section 3 we consider normal pseudo-BCI-algebras, that is, pseudo-BCI-algebras X, which are the sum of their pseudo-BCK-part $K(X)$ and p-semisimple part

2010 *Mathematics Subject Classification.* 03G25, 06F35.

Key words and phrases. Pseudo-BCI-algebra, normal pseudo-BCI-algebra, extension of pseudo-BCI-algebra.

$M(X)$. We illustrate this notion by interesting examples and give some characterizations of it. In this section we also construct a new pseudo-BCI-algebra being the sum of a pseudo-BCK-algebra and a p-semisimple pseudo-BCI-algebra (Theorem 3.4). Finally, in Section 4 we study extensions of pseudo-BCI-algebras.

2. Preliminaries. A *pseudo-BCI-algebra* is a structure $(X; \leq, \rightarrow, \rightsquigarrow, 1)$, where $\leq$ is a binary relation on a set X, $\rightarrow$ and $\rightsquigarrow$ are binary operations on X and 1 is an element of X such that for all $x, y, z \in X$, we have

(a1) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $\quad x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,
(a2) $x \leq (x \rightarrow y) \rightsquigarrow y$, $\quad x \leq (x \rightsquigarrow y) \rightarrow y$,
(a3) $x \leq x$,
(a4) if $x \leq y$ and $y \leq x$, then $x = y$,
(a5) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$.

It is obvious that any pseudo-BCI-algebra $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ can be regarded as a universal algebra $(X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$. Note that every pseudo-BCI-algebra satisfying $x \rightarrow y = x \rightsquigarrow y$ for all $x, y \in X$ is a BCI-algebra.

Every pseudo-BCI-algebra satisfying $x \leq 1$ for all $x \in X$ is a pseudo-BCK-algebra. A pseudo-BCI-algebra which is not a pseudo-BCK-algebra will be called *proper*.

Throughout this paper we will often use X to denote a pseudo-BCI-algebra. Any pseudo-BCI-algebra X satisfies the following, for all $x, y, z \in X$,

(b1) if $1 \leq x$, then $x = 1$,
(b2) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,
(b3) if $x \leq y$ and $y \leq z$, then $x \leq z$,
(b4) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$,
(b5) $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$,
(b6) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$, $\quad x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,
(b7) if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,
(b8) $1 \rightarrow x = 1 \rightsquigarrow x = x$,
(b9) $((x \rightarrow y) \rightsquigarrow y) \rightarrow y = x \rightarrow y$, $\quad ((x \rightsquigarrow y) \rightarrow y) \rightsquigarrow y = x \rightsquigarrow y$,
(b10) $x \rightarrow y \leq (y \rightarrow x) \rightsquigarrow 1$, $\quad x \rightsquigarrow y \leq (y \rightsquigarrow x) \rightarrow 1$,
(b11) $(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \rightsquigarrow (y \rightsquigarrow 1)$, $\quad (x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \rightarrow (y \rightarrow 1)$,
(b12) $x \rightarrow 1 = x \rightsquigarrow 1$.

If $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCI-algebra, then, by (a3), (a4), (b3) and (b1), $(X; \leq)$ is a poset with 1 as a maximal element. Note that a pseudo-BCI-algebra has also other maximal elements.

Proposition 2.1 ([4]). *The structure $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCI-algebra if and only if the algebra $(X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ satisfies the following identities and quasi-identity:*

(i) $(x \to y) \rightsquigarrow [(y \to z) \rightsquigarrow (x \to z)] = 1$,
(ii) $(x \rightsquigarrow y) \to [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1$,
(iii) $1 \to x = x$,
(iv) $1 \rightsquigarrow x = x$,
(v) $x \to y = 1 \;\;\&\;\; y \to x = 1 \;\Rightarrow\; x = y$.

Example 2.2 ([4]). Let $X = \{a, b, c, d, e, f, 1\}$ and define binary operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	a	b	c	d	e	f	1
a	1	d	e	b	c	a	a
b	c	1	a	e	d	b	b
c	e	a	1	c	b	d	d
d	b	e	d	1	a	c	c
e	d	c	b	a	1	e	e
f	a	b	c	d	e	1	1
1	a	b	c	d	e	f	1

$\rightsquigarrow$	a	b	c	d	e	f	1
a	1	c	b	e	d	a	a
b	d	1	e	a	c	b	b
c	b	e	1	c	a	d	d
d	e	a	d	1	b	c	c
e	c	d	a	b	1	e	e
f	a	b	c	d	e	1	1
1	a	b	c	d	e	f	1

Then $(X; \to, \rightsquigarrow, 1)$ is a (proper) pseudo-BCI-algebra. Observe that it is not a pseudo-BCK-algebra because $a \nleq 1$.

Example 2.3 ([9]). Let $Y_1 = (-\infty, 0]$ and let $\leq$ be the usual order on Y_1. Define binary operations $\to$ and $\rightsquigarrow$ on Y_1 by

$$x \to y = \begin{cases} 0 & \text{if} \quad x \leq y, \\ \frac{2y}{\pi} \arctan(\ln(\frac{y}{x})) & \text{if} \quad y < x, \end{cases}$$
$$x \rightsquigarrow y = \begin{cases} 0 & \text{if} \quad x \leq y, \\ y e^{-\tan(\frac{\pi x}{2y})} & \text{if} \quad y < x \end{cases}$$

for all $x, y \in Y_1$. Then $(Y_1; \leq, \to, \rightsquigarrow, 0)$ is a pseudo-BCK-algebra, and hence it is a nonproper pseudo-BCI-algebra.

Example 2.4 ([3]). Let $Y_2 = \mathbb{R}^2$ and define binary operations $\to$ and $\rightsquigarrow$ and a binary relation $\leq$ on Y_2 by

$$(x_1, y_1) \to (x_2, y_2) = (x_2 - x_1, (y_2 - y_1)e^{-x_1}),$$
$$(x_1, y_1) \rightsquigarrow (x_2, y_2) = (x_2 - x_1, y_2 - y_1 e^{x_2 - x_1}),$$

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow (x_1, y_1) \to (x_2, y_2) = (0, 0) = (x_1, y_1) \rightsquigarrow (x_2, y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in Y_2$. Then $(Y_2; \leq, \to, \rightsquigarrow, (0, 0))$ is a proper pseudo-BCI-algebra. Notice that Y_2 is not a pseudo-BCK-algebra because there exists $(x, y) = (1, 1) \in Y_2$ such that $(x, y) \nleq (0, 0)$.

Example 2.5 ([3]). Let Y be the direct product of pseudo-BCI-algebras Y_1 and Y_2 from Examples 2.3 and 2.4, respectively. Then Y is a proper pseudo-BCI-algebra, where $Y = (-\infty, 0] \times \mathbb{R}^2$ and binary operations $\to$ and

$\rightsquigarrow$ and binary relation $\leq$ are defined on Y by

$$(x_1, y_1, z_1) \to (x_2, y_2, z_2) = \\ = \begin{cases} (0, y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_1 \leq x_2, \\ (\frac{2x_2}{\pi} \arctan(\ln(\frac{x_2}{x_1})), y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_2 < x_1, \end{cases}$$

$$(x_1, y_1, z_1) \rightsquigarrow (x_2, y_2, z_2) = \\ = \begin{cases} (0, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_1 \leq x_2, \\ (x_2 e^{-\tan(\frac{\pi x_1}{2x_2})}, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_2 < x_1, \end{cases}$$

$$(x_1, y_1, z_1) \leq (x_2, y_2, z_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 = y_2 \text{ and } z_1 = z_2.$$

Notice that Y is not a pseudo-BCK-algebra because there exists $(x, y, z) = (0, 1, 1) \in Y$ such that $(x, y, z) \nleq (0, 0, 0)$.

For any pseudo-BCI-algebra $(X; \to, \rightsquigarrow, 1)$ the set

$$K(X) = \{x \in X : x \leq 1\}$$

is a subalgebra of X (called pseudo-BCK-part of X). Then $(K(X); \to, \rightsquigarrow, 1)$ is a pseudo-BCK-algebra. Note that a pseudo-BCI-algebra X is a pseudo-BCK-algebra if and only if $X = K(X)$.

It is easily seen that for the pseudo-BCI-algebras X, Y_1, Y_2 and Y from Examples 2.2, 2.3, 2.4 and 2.5, respectively, we have $K(X) = \{f, 1\}$, $K(Y_1) = Y_1$, $K(Y_2) = \{(0, 0)\}$ and $K(Y) = \{(x, 0, 0) : x \leq 0\}$.

We will denote by $M(X)$ the set of all maximal elements of X and call it the p-semisimple part of X. Obviously, $1 \in M(X)$. Notice that $M(X) \cap K(X) = \{1\}$. Indeed, if $a \in M(X) \cap K(X)$, then $a \leq 1$ and, by the fact that a is maximal, $a = 1$. Moreover, observe that 1 is the only maximal element of a pseudo-BCK-algebra. Therefore, for a pseudo-BCK-algebra X, $M(X) = \{1\}$. In [2] and [3] there is shown that $M(X) = \{x \in X : x = (x \to 1) \to 1\}$ and it is a subalgebra of X.

Observe that for the pseudo-BCI-algebras X, Y_1, Y_2 and Y from Examples 2.2, 2.3, 2.4 and 2.5, respectively, we have $M(X) = \{a, b, c, d, e, 1\}$, $M(Y_1) = \{0\}$, $M(Y_2) = Y_2$ and $M(Y) = \{(0, y, z) : y, z \in \mathbb{R}\}$.

Proposition 2.6. *Let X be a pseudo-BCI-algebra. Then*

$$M(X) = \{x \to 1 : x \in X\}.$$

Proof. We know that

$$M(X) = \{x \in X : x = (x \to 1) \to 1\}.$$

Since, by (b9) and (b12), for any $x \in X$,

$$x \to 1 = ((x \to 1) \to 1) \to 1,$$

we get that $x \to 1 \in M(X)$ for any $x \in X$. Hence,

$$\{x \to 1 : x \in X\} \subseteq M(X).$$

Now, let $a \in M(X)$. Then, $a = (a \to 1) \to 1$. Putting $x = a \to 1 \in X$ we obtain that $a = x \to 1$ for some $x \in X$ and also

$$M(X) \subseteq \{x \to 1 : x \in X\}.$$

Therefore, $M(X) = \{x \to 1 : x \in X\}$. □

Let X be a pseudo-BCI-algebra. For any $a \in X$ we define a subset $V(a)$ of X as follows

$$V(a) = \{x \in X : x \leq a\}.$$

Note that $V(a)$ is non-empty, because $a \leq a$ gives $a \in V(a)$. Notice also that $V(a) \subseteq V(b)$ for any $a, b \in X$ such that $a \leq b$.

If $a \in M(X)$, then the set $V(a)$ is called a *branch* of X determined by element a. The following facts are proved in [3]: (1) branches determined by different elements are disjoint, (2) a pseudo-BCI-algebra is a set-theoretic union of branches, (3) comparable elements are in the same branch.

The pseudo-BCI-algebra Y_1 from Example 2.3 has only one branch (as the pseudo-BCK-algebra) and the pseudo-BCI-algebra X from Example 2.2 has six branches: $V(a) = \{a\}$, $V(b) = \{b\}$, $V(c) = \{c\}$, $V(d) = \{d\}$, $V(e) = \{e\}$ and $V(1) = \{f, 1\}$. Every $\{(x, y)\}$ is a branch of the pseudo-BCI-algebra Y_2 from Example 2.4, where $(x, y) \in Y_2$. For the pseudo-BCI-algebra Y from Example 2.5 the sets $V((0, a_1, a_2)) = \{(x, a_1, a_2) \in Y : x \leq 0\}$, where $(0, a_1, a_2) \in M(X)$, are branches of Y.

Proposition 2.7 ([2]). *Let X be a pseudo-BCI-algebra and let $x \in X$ and $a, b \in M(X)$. If $x \in V(a)$, then $x \to b = a \to b$ and $x \rightsquigarrow b = a \rightsquigarrow b$.*

Proposition 2.8 ([2]). *Let X be a pseudo-BCI-algebra and let $x, y \in X$. The following are equivalent:*

(i) *x and y belong to the same branch of X,*
(ii) *$x \to y \in K(X)$,*
(iii) *$x \rightsquigarrow y \in K(X)$.*

Proposition 2.9 ([3]). *Let X be a pseudo-BCI-algebra and let $x, y \in X$. If x and y belong to the same branch of X, then $x \to 1 = x \rightsquigarrow 1 = y \to 1 = y \rightsquigarrow 1$.*

We have the following proposition.

Proposition 2.10. *Let X be a pseudo-BCI-algebra and let $x, y \in X$. The following are equivalent:*

(i) *x and y belong to the same branch of X,*
(ii) *$x \to y \in K(X)$,*
(iii) *$x \rightsquigarrow y \in K(X)$,*

(iv) $x \to 1 = x \rightsquigarrow 1 = y \to 1 = y \rightsquigarrow 1$.

Proof. Let $x, y \in X$. By Propositions 2.8 and 2.9 and (b12) it is sufficient to prove that if $x \to 1 = y \to 1$, then $x \to y \in K(X)$, that is, (iv) $\Rightarrow$ (ii). Assume that $x \to 1 = y \to 1$. Then, by (b11) and (b12), we have $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \to 1) = 1$, which means that $x \to y \leq 1$. Hence, $x \to y \in K(X)$ and the proof is complete. □

We also have the following proposition.

Proposition 2.11. *Let X be a pseudo-BCI-algebra and let $x, y \in X$. The following are equivalent:*

(i) *x and y belong to the same branch of X,*
(ii) *$x \to a = y \to a$ for all $a \in M(X)$,*
(ii') *$x \rightsquigarrow a = y \rightsquigarrow a$ for all $a \in M(X)$,*
(iii) *$x \to a \leq y \to a$ for all $a \in M(X)$,*
(iii') *$x \rightsquigarrow a \leq y \rightsquigarrow a$ for all $a \in M(X)$.*

Proof. (i) $\Rightarrow$ (ii): Assume that $x, y \in V(b)$ for some $b \in M(X)$. Then for any $a \in M(X)$, by Proposition 2.7, we get $x \to a = b \to a = y \to a$, that is, (ii) holds.

(ii) $\Rightarrow$ (i): If $x \to a = y \to a$ for all $a \in M(X)$, then putting $a = 1$ we get $x \to 1 = y \to 1$. Now, by Proposition 2.10, we obtain (i).

(ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (ii): Let $x \to a \leq y \to a$ for all $a \in M(X)$. Then, since $x \to a \in M(X)$ by Proposition 2.7, we have that $x \to a = y \to a$ for all $a \in M(X)$.

Similarly, we can prove the equivalences (i) $\Leftrightarrow$ (ii') $\Leftrightarrow$ (iii'). □

Proposition 2.12. *Let X be a pseudo-BCI-algebra and let $x \in X$ and $a \in M(X)$. Then the following are equivalent:*

(i) *$x \in V(a)$,*
(ii) *$x \to b = a \to b$ for all $b \in M(X)$,*
(iii) *$x \rightsquigarrow b = a \rightsquigarrow b$ for all $b \in M(X)$.*

Proof. (i) $\Rightarrow$ (ii): Follows by Proposition 2.7.

(ii) $\Rightarrow$ (i): Let $x \in X$ and $a \in M(X)$. Assume that $x \to b = a \to b$ for all $b \in M(X)$. Putting $b = 1$ we get $x \to 1 = a \to 1$. Hence, by Proposition 2.10, x and a are in the same branch of X, that is, $x \in V(a)$.

(i) $\Leftrightarrow$ (iii): Analogous. □

Let $(X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then X is *p-semisimple* if it satisfies for all $x \in X$,

$$\text{if } x \leq 1, \text{ then } x = 1.$$

Note that if X is a p-semisimple pseudo-BCI-algebra, then $K(X) = \{1\}$. Hence, if X is a p-semisimple pseudo-BCK-algebra, then $X = \{1\}$. Moreover, as it is proved in [3], $M(X)$ is a p-semisimple pseudo-BCI-subalgebra of X and X is p-semisimple if and only if $X = M(X)$.

It is not difficult to see that the pseudo-BCI-algebras X, Y_1 and Y from Examples 2.2, 2.3 and 2.5, respectively, are not p-semisimple, and the pseudo-BCI-algebra Y_2 from Example 2.4 is a p-semisimple algebra.

Proposition 2.13 ([3]). *Let X be a pseudo-BCI-algebra. Then, for all $a, b, x, y \in X$, the following are equivalent:*

(i) *X is p-semisimple,*
(ii) $(x \to y) \rightsquigarrow y = x = (x \rightsquigarrow y) \to y$,
(iii) $(x \to 1) \rightsquigarrow 1 = x = (x \rightsquigarrow 1) \to 1$,
(iv) *if $x \to a = x \to b$, then $a = b$,*
(v) *if $x \rightsquigarrow a = x \rightsquigarrow b$, then $a = b$,*
(vi) *if $a \to x = b \to x$, then $a = b$,*
(vii) *if $a \rightsquigarrow x = b \rightsquigarrow x$, then $a = b$.*

3. Normal pseudo-BCI-algebras. A pseudo-BCI-algebra X is called *normal* if $X = K(X) \cup M(X)$. Otherwise, it is called *non-normal*.

Remark. Every pseudo-BCK-algebra and every p-semisimple pseudo-BCI-algebra are normal.

A pseudo-BCI-algebra X is called *strongly normal* if X is normal and $K(X) \neq \{1\}$ and $M(X) \neq \{1\}$.

Example 3.1. It is easy to see that the pseudo-BCI-algebra X from Example 2.2 is strongly normal, and the pseudo-BCI-algebra Y from Example 2.5 is non-normal.

Theorem 3.2. *Let X be a pseudo-BCI-algebra. Then the following are equivalent:*

(i) *X is normal,*
(ii) $((x \to 1) \to 1) \to x \in \{x, 1\}$ *for any* $x \in X$,
(iii) $((x \to 1) \to 1) \rightsquigarrow x \in \{x, 1\}$ *for any* $x \in X$.

Proof. (i) $\Rightarrow$ (ii): Let X be normal. Then $X = K(X) \cup M(X)$. Let $x \in X$. If $x \in K(X)$, then

$$((x \to 1) \to 1) \to x = 1 \to x = x \in \{x, 1\}.$$

If $x \in M(X)$, then

$$((x \to 1) \to 1) \to x = x \to x = 1 \in \{x, 1\}.$$

(ii) $\Rightarrow$ (i): Let $((x \to 1) \to 1) \to x \in \{x, 1\}$ for any $x \in X$. Take $z \in X$. If $((z \to 1) \to 1) \to z = z$, then, by (b9), b(11) and (b12),

$$\begin{aligned} z \to 1 &= (((z \to 1) \to 1) \to z) \to 1 \\ &= (((z \to 1) \to 1) \to 1) \rightsquigarrow (z \to 1) \\ &= (z \to 1) \rightsquigarrow (z \to 1) \\ &= 1 \end{aligned}$$

Hence, $z \leq 1$, that is, $z \in K(X)$. If $((z \to 1) \to 1) \to z = 1$, then, $(z \to 1) \to 1 \leq z$. Hence and by (a2) and (b12),

$$z = (z \to 1) \to 1,$$

which means that $z \in M(X)$. Hence, $X = K(X) \cup M(X)$, that is, it is normal.

(i) $\Leftrightarrow$ (iii): Analogously. □

In next theorem we construct some strongly normal pseudo-BCI-algebra. But first, we prove the following lemma.

Lemma 3.3. *Let X be a pseudo-BCI-algebra. Then*

(i) *for any $x \in X$ and $y \in K(X)$,*

$$(x \to y) \to (x \to 1) = 1 = ((x \to 1) \to (x \to y)) \to 1,$$
$$(x \to y) \rightsquigarrow (x \to 1) = 1 = ((x \to 1) \rightsquigarrow (x \to y)) \to 1,$$
$$(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow 1) = 1 = ((x \rightsquigarrow 1) \rightsquigarrow (x \rightsquigarrow y)) \to 1,$$
$$(x \rightsquigarrow y) \to (x \rightsquigarrow 1) = 1 = ((x \rightsquigarrow 1) \to (x \rightsquigarrow y)) \to 1,$$

(ii) *for any $x \in K(X)$ and $a \in M(X)$,*

$$x \to a = a = x \rightsquigarrow a = (a \to x) \to 1 = (a \rightsquigarrow x) \to 1,$$

(iii) *if $X = K(X) \cup M(X)$, then $a \to x = a \to 1 = a \rightsquigarrow x$ for any $a \in M(X)\backslash\{1\}$ and $x \in K(X)$.*

Proof. (i) Let $x \in X$ and $y \in K(X)$. By (b1) and (b6), $(x \to y) \to (x \to 1) = 1$. Then, by (b10), $1 = (x \to y) \to (x \to 1) \leq ((x \to 1) \to (x \to y)) \to 1$. Hence, by (b1),

$$(x \to y) \to (x \to 1) = 1 = ((x \to 1) \to (x \to y)) \to 1.$$

Next, by (b4), (b11) and (b12) we have

$$\begin{aligned}(x \to y) \rightsquigarrow (x \to 1) &= x \to ((x \to y) \rightsquigarrow 1)\\ &= x \to ((x \to 1) \rightsquigarrow (y \to 1))\\ &= x \to ((x \to 1) \rightsquigarrow 1)\\ &= (x \to 1) \rightsquigarrow (x \to 1)\\ &= 1.\end{aligned}$$

Now, it is easy to see that

$$(x \to y) \rightsquigarrow (x \to 1) = 1 = ((x \to 1) \rightsquigarrow (x \to y)) \to 1.$$

Similarly, we can prove other equations of (i).

(ii) Let $x \in K(X)$ and $a \in M(X)$. From Proposition 2.12 we immediately have that

$$x \to a = a = x \rightsquigarrow a.$$

Moreover, by (b10) and (b12), $a = x \to a \leq ((a \to x) \to 1$ and $a = x \rightsquigarrow a \leq ((a \rightsquigarrow x) \to 1$. Since $a \in M(X)$, we get (ii).

(iii) Let $X = K(X) \cup M(X)$, $a \in M(X)\backslash\{1\}$ and $x \in K(X)$. By (ii), $(a \to x) \to 1 = a \neq 1$. Hence, $a \to x \notin K(X)$, that is, $a \to x \in M(X)\backslash\{1\}$. Then, $(a \to 1) \to (a \to x) \in M(X)$. But, by (i), $(a \to x) \to (a \to 1) = 1 = ((a \to 1) \to (a \to x)) \to 1$. Thus, $a \to x \leq a \to 1$ and $(a \to 1) \to (a \to x) = 1$, that is, also $a \to 1 \leq a \to x$. Therefore, $a \to x = a \to 1$. Similarly, we prove that $a \rightsquigarrow x = a \to 1$. □

Remark. Note that the assumption $X = K(X) \cup M(X)$ in Lemma 3.3 (iii) is valid. Indeed, let Y be the pseudo-BCI-algebra from Example 2.5. We know that $K(Y) = \{(x, 0, 0) : x \leq 0\}$ and $M(Y) = \{(0, y, z) : y, z \in \mathbb{R}\}$. Then for $x < 0$ and $a_1, a_2 \in \mathbb{R}$ we have

$$\begin{aligned}(0, a_1, a_2) \to (x, 0, 0) = (0, a_1, a_2) \rightsquigarrow (x, 0, 0) &= (x, -a_1, -a_2 e^{-a_1}) \\ &\neq (0, a_1, a_2) \to (0, 0, 0) \\ &= (0, -a_1, -a_2 e^{-a_1}).\end{aligned}$$

Theorem 3.4. *Let Y be a pseudo-BCK-algebra, Z be a (proper) p-semisimple pseudo-BCI-algebra and $Y \cap Z = \{1\}$. Then there exists a unique pseudo-BCI-algebra X such that $X = Y \cup Z$, $K(X) = Y$ and $M(X) = Z$.*

Proof. First, the operations on Y and Z we denote by the same symbols $\to$ and $\rightsquigarrow$. Define on $X = Y \cup Z$ binary operations $\mapsto$ and $\curvearrowright$ as follows

$$x \mapsto y = \begin{cases} x \to y & \text{if } x, y \in Y \text{ or } x, y \in Z, \\ y & \text{if } x \in Y \text{ and } y \in Z\backslash\{1\}, \\ x \to 1 & \text{if } x \in Z\backslash\{1\} \text{ and } y \in Y \end{cases}$$

and

$$x \curvearrowright y = \begin{cases} x \rightsquigarrow y & \text{if } x, y \in Y \text{ or } x, y \in Z, \\ y & \text{if } x \in Y \text{ and } y \in Z\backslash\{1\}, \\ x \rightsquigarrow 1 & \text{if } x \in Z\backslash\{1\} \text{ and } y \in Y. \end{cases}$$

We show that $(X; \mapsto, \curvearrowright, 1)$ is a pseudo-BCI-algebra. We check the conditions (i)–(v) of Proposition 2.1. Since Y and Z are pseudo-BCI-algebras, we only need checking these conditions for the elements which are not all in Y and not all in Z. Particularly, (iii) and (iv) are satisfied. Now, we prove (v). Let $x \in Y$ and $y \in Z$. Assume that $x \mapsto y = 1 = y \mapsto x$. Then, $y = x \mapsto y = 1$. This means that $x = 1 \mapsto x = 1$, that is, $x = y = 1$. Thus, (v) is satisfied. Next, we show the identity (i). Let $x, x_1, x_2 \in Y$ and $y, y_1, y_2 \in Z$. Then

(1) $(x \mapsto y_1) \curvearrowright [(y_1 \mapsto y_2) \curvearrowright (x \mapsto y_2)] = y_1 \rightsquigarrow [(y_1 \to y_2) \rightsquigarrow y_2] = y_1 \rightsquigarrow y_1 = 1$,

(2) $(y_1 \mapsto x) \curvearrowright [(x \mapsto y_2) \curvearrowright (y_1 \mapsto y_2)] = (y_1 \to 1) \rightsquigarrow [y_2 \rightsquigarrow (y_1 \to y_2)] = (y_1 \to 1) \rightsquigarrow (y_1 \to 1) = 1$,

(3) $(y_1 \mapsto y_2) \curvearrowright [(y_2 \mapsto x) \curvearrowright (y_1 \mapsto x)] = (y_1 \to y_2) \rightsquigarrow [(y_2 \to 1) \rightsquigarrow (y_1 \to 1)] = 1$,
(4) $(y \mapsto x_1) \curvearrowright [(x_1 \mapsto x_2) \curvearrowright (y \mapsto x_2)] = (y \to 1) \curvearrowright [(x_1 \to x_2) \curvearrowright (y \to 1)] = (y \to 1) \rightsquigarrow (y \to 1) = 1$,
(5) $(x_1 \mapsto y) \curvearrowright [(y \mapsto x_2) \curvearrowright (x_1 \mapsto x_2)] = y \curvearrowright [(y \to 1) \curvearrowright (x_1 \to x_2)] = y \curvearrowright [(y \to 1) \rightsquigarrow 1] = y \rightsquigarrow y = 1$,
(6) $(x_1 \mapsto x_2) \curvearrowright [(x_2 \mapsto y) \curvearrowright (x_1 \mapsto y)] = (x_1 \to x_2) \curvearrowright (y \rightsquigarrow y) = y \rightsquigarrow y = 1$.

Thus, (i) is also satisfied. Similarly we can prove (ii). Therefore, $(X; \mapsto, \curvearrowright, 1)$ is a pseudo-BCI-algebra.

Now, note that $x \mapsto 1 = x \to 1$ for every $x \in X$. This means that $x \mapsto 1 = 1$ if and only if $x \to 1 = 1$, and $(x \mapsto 1) \mapsto 1 = x$ if and only if $(x \to 1) \to 1 = x$. Hence, $K(X) = Y$ and $M(X) = Z$.

Finally, we show uniqueness of pseudo-BCI-algebra $(X; \mapsto, \curvearrowright, 1)$. Let $(X; \rightarrowtail, \looparrowright, 1)$ be a pseudo-BCI-algebra such that $X = Y \cup Z$, $K(X) = Y$ and $M(X) = Z$. If $x, y \in Y$ or $x, y \in Z$, then

$$x \rightarrowtail y = x \to y = x \mapsto y \quad \text{and} \quad x \looparrowright y = x \rightsquigarrow y = x \curvearrowright y.$$

If $x \in Y$ and $y \in Z \backslash \{1\}$, then, by Lemma 3.3,

$$x \rightarrowtail y = y = x \mapsto y \quad \text{and} \quad x \looparrowright y = y = x \curvearrowright y.$$

If $x \in Z \backslash \{1\}$ and $y \in Y$, then, again by Lemma 3.3,

$$x \rightarrowtail y = x \rightarrowtail 1 = x \to 1 = x \mapsto y$$

and

$$x \looparrowright y = x \looparrowright 1 = x \rightsquigarrow 1 = x \curvearrowright y.$$

Therefore, $(X; \rightarrowtail, \looparrowright, 1) = (X; \mapsto, \curvearrowright, 1)$. □

Remark. Notice that a pseudo-BCI-algebra X constructed in Theorem 3.4 is strongly normal.

Example 3.5. Take the following pseudo-BCK-algebra $Y = \{\alpha, \beta, \gamma, 1\}$ equipped with the operations $\to$ and $\rightsquigarrow$ given by the following tables (see [6]):

$\to$	α	β	γ	1
α	1	1	1	1
β	β	1	1	1
γ	β	β	1	1
1	α	β	γ	1

$\rightsquigarrow$	α	β	γ	1
α	1	1	1	1
β	γ	1	1	1
γ	α	β	1	1
1	α	β	γ	1

and the following p-semisimple pseudo-BCI-algebra $Z = \{a, b, c, d, e, 1\}$ equipped with the operations $\rightarrow$ and $\rightsquigarrow$ given by the following tables (see [4]):

$\rightarrow$	a	b	c	d	e	1
a	1	d	e	b	c	a
b	c	1	a	e	d	b
c	e	a	1	c	b	d
d	b	e	d	1	a	c
e	d	c	b	a	1	e
1	a	b	c	d	e	1

$\rightsquigarrow$	a	b	c	d	e	1
a	1	c	b	e	d	a
b	d	1	e	a	c	b
c	b	e	1	c	a	d
d	e	a	d	1	b	c
e	c	d	a	b	1	e
1	a	b	c	d	e	1

Then, using Theorem 3.4, we can construct the new pseudo-BCI-algebra $(X; \mapsto, \curvearrowright, 1)$ such that $X = Y \cup Z$ and the operations $\mapsto$ and $\curvearrowright$ are as follows:

$\mapsto$	α	β	γ	a	b	c	d	e	1
α	1	1	1	a	b	c	d	e	1
β	β	1	1	a	b	c	d	e	1
γ	β	β	1	a	b	c	d	e	1
a	a	a	a	1	d	e	b	c	a
b	b	b	b	c	1	a	e	d	b
c	d	d	d	e	a	1	c	b	d
d	c	c	c	b	e	d	1	a	c
e	e	e	e	d	c	b	a	1	e
1	α	β	γ	a	b	c	d	e	1

and

$\curvearrowright$	α	β	γ	a	b	c	d	e	1
α	1	1	1	a	b	c	d	e	1
β	γ	1	1	a	b	c	d	e	1
γ	α	β	1	a	b	c	d	e	1
a	a	a	a	1	c	b	e	d	a
b	b	b	b	d	1	e	a	c	b
c	d	d	d	b	e	1	c	a	d
d	c	c	c	e	a	d	1	b	c
e	e	e	e	c	d	a	b	1	e
1	α	β	γ	a	b	c	d	e	1

Obviously, $K(X) = Y$ and $M(X) = Z$, that is, X is strongly normal.

4. Extensions of pseudo-BCI-algebras. Let X and X^* be pseudo-BCI-algebras. If X is a subalgebra of X^*, then X^* is called an *extension* of X. If X^* is p-semisimple (respectively, strongly normal, non-normal), then X^* is called a *p-semisimple* (respectively, *strongly normal*, *non-normal*) *extension* of X. If $|X^* \backslash X| = 1$, then X^* is called a *simple extension* of X.

First, we show some simple lemma. Consider the map $p : X \to X$ such that

$$p(x) = x \to 1$$

for all $x \in X$. Obviously, $p(x) = x \rightsquigarrow 1$ for all $x \in X$. Note that $Im(p) = M(X)$, $Ker(p) = K(X)$ and if X is p-semisimple, then p is surjective.

Lemma 4.1. *Let X be a p-semisimple pseudo-BCI-algebra. Then, for all $a \in X$, maps $f_a^{\rightarrow}, f_a^{\rightsquigarrow}, g_a^{\rightarrow}, g_a^{\rightsquigarrow} : X \to X$ such that*

$$\begin{aligned} f_a^{\rightarrow}(x) &= x \to a, \\ f_a^{\rightsquigarrow}(x) &= x \rightsquigarrow a, \\ g_a^{\rightarrow}(x) &= a \to x, \\ g_a^{\rightsquigarrow}(x) &= a \rightsquigarrow x \end{aligned}$$

for all $x \in X$, are injective. Moreover, $g_a^{\rightarrow}$ and $g_a^{\rightsquigarrow}$ are also surjective.

Proof. Since X is p-semisimple, immediately by Proposition 2.13, $f_a^{\rightarrow}$, $f_a^{\rightsquigarrow}$, $g_a^{\rightarrow}$, $g_a^{\rightsquigarrow}$ are injective. Moreover, for all $x \in X$, by (b4) we have

$$\begin{aligned} (g_a^{\rightarrow} \circ f_a^{\rightsquigarrow})(x) &= g_a^{\rightarrow}(x \rightsquigarrow a) = a \to (x \rightsquigarrow a) \\ &= x \rightsquigarrow (a \to a) = x \rightsquigarrow 1 \\ &= p(x) \end{aligned}$$

and

$$\begin{aligned} (g_a^{\rightsquigarrow} \circ f_a^{\rightarrow})(x) &= g_a^{\rightsquigarrow}(x \to a) = a \rightsquigarrow (x \to a) \\ &= x \to (a \rightsquigarrow a) = x \to 1 \\ &= p(x) \end{aligned}$$

Hence, since p is surjective, maps $g_a^{\rightarrow}$ and $g_a^{\rightsquigarrow}$ are surjective. □

Remark. Note that $g_a^{\rightarrow} \circ f_a^{\rightsquigarrow} = g_a^{\rightsquigarrow} \circ f_a^{\rightarrow}$ and the map p can be decomposed into an injection and a bijection.

Theorem 4.2. *Let X be a p-semisimple pseudo-BCI-algebra. Then*

(i) *there is no p-semisimple simple extension of X if $|X| \geq 2$,*
(ii) *there is a unique strongly normal simple extension of X,*
(iii) *there is no non-normal simple extension of X.*

Proof. (i) Let X be a p-semisimple pseudo-BCI-algebra and $|X| \geq 2$. Assume that $X^* = X \cup \{x_0\}$ is a p-semisimple extension of X. Since $|X| \geq 2$, we can take $x \in X \backslash \{1\}$. Now, take the map $g_x^{\rightarrow} : X^* \to X^*$. By Lemma 4.1 we have $g_x^{\rightarrow}(X^*) = X^*$ and $g_x^{\rightarrow}(X) = X$. Note that $g_x^{\rightarrow}(x_0) \in X$. Indeed, if $g_x^{\rightarrow}(x_0) \in X^* \backslash X = \{x_0\}$, then $x \to x_0 = x_0 = 1 \to x_0$ and by Proposition 2.13, $x = 1$, which is impossible. Hence, $g_x^{\rightarrow}(x_0) \in X$. Thus, $g_x^{\rightarrow}(X^*) = g_x^{\rightarrow}(X) \cup \{g_x^{\rightarrow}(x_0)\} = X$ and we have a contradiction.

(ii) First, there is a unique (pseudo-)BCK-algebra $B_0 = \{0, 1\}$ in which the operation $\to$ is as follows

$\to$	0	1
0	1	1
1	0	1

Now, it is sufficient to take a pseudo-BCI-algebra $X^* = B_0 \cup X$ as in Theorem 3.4. Obviously, X^* is the unique strongly normal simple extension of X.

(iii) It follows from (i) and the fact that for any pseudo-BCI-algebra Y we have $K(Y) = \{1\}$ if and only if $M(Y) = Y$. □

Corollary 4.3. *If X is a p-semisimple pseudo-BCI-algebra such that $|X| \geq 3$, then X is not a simple extension of any pseudo-BCI-algebra.*

For arbitrary pseudo-BCI-algebras we have the following theorem.

Theorem 4.4 ([4]). *Any pseudo-BCI-algebra has a simple extension.*

Remark. Note that for a pseudo-BCI-algebra X a new element of its simple extension X^* constructed in [4] belongs to $K(X)$. This means that if X is strongly normal (respectively, non-normal), then also X^* is strongly normal (respectively, non-normal).

References

[1] Dudek, W. A., Jun, Y. B., *Pseudo-BCI algebras*, East Asian Math. J. **24** (2008), 187–190.

[2] Dymek, G., *Atoms and ideals of pseudo-BCI-algebras*, Comment. Math. **52** (2012), 73–90.

[3] Dymek, G., *p-semisimple pseudo-BCI-algebras*, J. Mult.-Valued Logic Soft Comput. **19** (2012), 461–474.

[4] Dymek, G., *On compatible deductive systems of pseudo-BCI-algebras*, J. Mult.-Valued Logic Soft Comput. **22** (2014), 167–187.

[5] Dymek, G., Kozanecka-Dymek, A., *Pseudo-BCI-logic*, Bull. Sect. Logic Univ. Łódż **42** (2013), 33–42.

[6] Halaš, R., Kühr, J., *Deductive systems and annihilators of pseudo-BCK-algebras*, Ital. J. Pure Appl. Math. **25** (2009).

[7] Iorgulescu, A., *Algebras of Logic as BCK Algebras*, Editura ASE, Bucharest, 2008.

[8] Iséki, K., *An algebra related with a propositional calculus*, Proc. Japan Acad. **42** (1966), 26–29.

[9] Jun, Y. B., Kim, H. S., Neggers, J., *On pseudo-BCI ideals of pseudo BCI-algebras*, Mat. Vesnik **58** (2006), 39–46.

Grzegorz Dymek
Institute of Mathematics and Computer Science
The John Paul II Catholic University of Lublin
Konstantynów 1H
20-708 Lublin
Poland
e-mail: gdymek@o2.pl

Received September 9, 2014